ELEMENTS

OF

ALGEBRA,

THEORETICAL AND PRACTICAL,

FOR THE

Use of Schools and Private Students:

CONTAINING THE

FUNDAMENTAL RULES, FRACTIONS, INVOLUTION AND EVOLUTION, SURDS, EQUATIONS OF ALL DEGREES, PROGRESSIONS, SERIES, LOGARITHMS AND THEIR APPLICATIONS, PROPERTIES OF NUMBERS, CONTINUED FRACTIONS AND THEIR USES, THE INDETERMINATE OR DIOPHANTINE ANALYSIS, PROBABILITIES, LIFE ANNUITIES, ETC.

WITH

NUMEROUS EXERCISES UNDER EACH HEAD, AND A LARGE COLLECTION OF

MISCELLANEOUS QUESTIONS.

BY ALEXANDER INGRAM,

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AND

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PREFACE.

The groundwork of the following Treatise belongs to the late Mr Ingram, than whom few men knew better what was suited to the capacities of the young. The public have already pronounced a favourable opinion on the Algebra contained in his Concise System of Mathematics; and in extending it to its present size every endeavour has been used to adhere as much as possible to the original plan. The chief object of that work was to combine theory and practice in such a manner that the principle of every rule should be thoroughly understood before proceeding to solve the exercises under the rule; and it is hoped that, at least in the higher departments, this has been fully attained. Usefulness and simplicity have been considered of far more importance than originality, and therefore free use has been made of the labours of our older authors for many of the exercises. With these, however, will be found interspersed many new and interesting questions; while care has been taken that every part of the work should be so clearly and simply expressed, that the most ordinary capacity can scarcely fail to comprehend it.

Although among the quadratic equations of the original work there were given several questions involving two unknown quantities, yet, as no specific rules were laid down for the solution of such equations, it has been thought expedient to treat these in a regular and systematic manner. The portion of the volume devoted to the properties of numbers may probably by some students be considered too lengthy for an elementary treatise; but by those who are fond of theoretical speculations, and alive to the intrinsic value of the subject, no such fault will be found. Continued fractions and their various applications have been

treated at considerable length, though not to a greater extent than their importance seemed to demand. The chief feature, however, to which the Publishers would draw attention, is the very full and complete exposition of the Indeterminate or Diophantine Analysis. Of all the branches of Algebra this is the most difficult and abstruse :—the various substitutions necessary to effect the solutions of the problems, the ingenuity and dexterity requisite in making the original assumptions, and the caution and judgment indispensable in evolving the final result, tend more to sharpen and invigorate the intellectual powers than the study of any other subject in the whole course of human investigation.

Great attention having been devoted to the selection of the exercises, it is confidently hoped that they will be found appropriate, and also of sufficient number and variety to initiate the student in the various rules; while the very large collection of miscellaneous questions at the end of the volume will afford a vast fund of profit and amusement to those who may undertake their solution, and will at the same time test their knowledge of the principles previously laid down.

Although every care has been taken to ensure accuracy, yet it is possible that errors may be found. These, however, when noticed, will be corrected; and in a second edition such alterations and additions will be made as may suggest themselves in teaching the work.

EDINBURGH, *August* 12, 1844.

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II. Reduce $\frac{749}{8}$, $\frac{3741}{17}$, $\frac{4638}{18}$, $\frac{2176}{21}$, $\frac{3876}{29}$, and $\frac{74896}{37}$, to mixed numbers. Ans. $93\frac{5}{8}$, $220\frac{1}{17}$, $257\frac{2}{3}$, $103\frac{13}{21}$, $133\frac{19}{29}$, and $2024\frac{8}{37}$.

III. Reduce $\frac{7}{\frac{3}{8}}$, $\frac{\frac{3}{4}}{7}$, $\frac{\frac{2}{3}}{\frac{7}{8}}$, $\frac{3\frac{1}{3}}{\frac{4}{7}}$, $\frac{\frac{2}{5}}{4\frac{1}{5}}$, and $\frac{7\frac{5}{9}}{12\frac{3}{5}}$, to simple fractions.

Ans. $\frac{56}{3}$, $\frac{3}{28}$, $\frac{16}{21}$, $\frac{35}{6}$, $\frac{2}{21}$, and $\frac{340}{567}$.

IV. Reduce $\frac{2}{7}$ of $\frac{3}{4}$ of $3\frac{1}{2}$, $\frac{5}{8}$ of $\frac{6}{7}$ of $4\frac{1}{5}$, $\frac{2}{5}$ of $\frac{5}{7}$ of $\frac{11}{15}$ of $4\frac{1}{2}$, and $\frac{3}{11}$ of $\frac{7}{12}$ of $\frac{4}{21}$ of $2\frac{3}{4}$, to simple fractions. Ans. $\frac{3}{4}$, $\frac{9}{4}$, $\frac{33}{35}$, and $\frac{1}{12}$.

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Ans. $\frac{12}{36}$, $\frac{30}{36}$, $\frac{16}{36}$, and $\frac{21}{36}$; $\frac{90}{120}$, $\frac{105}{120}$, $\frac{108}{120}$, and $\frac{88}{120}$; $\frac{42}{21}$, $\frac{7}{21}$, $\frac{12}{21}$, and $\frac{14}{21}$; $\frac{30}{60}$, $\frac{15}{60}$, $\frac{28}{60}$, and $\frac{24}{60}$.

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Ans. $1\frac{4913}{5544}$; $15\frac{2683}{3276}$; $5\frac{7}{18}$; $5\frac{4175}{4788}$.

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Ans. 4; $\frac{13}{1088}$; $37\frac{1}{8}$; $1\frac{67903}{333212}$.

PRAXIS OF DECIMAL FRACTIONS.

I. Add 2·71 + 34·176 + 347·897 + 2753·8172 + ·214 + ·000741 + 21·7104. Ans. 3160·525341.

Add 175 + ·175 + 31·007 + 352·86 + 7148·307 + 17·8 + 21 + ·00321. Ans. 7746·15221.

Add $\cdot 07\dot{6} + 1\cdot 21\dot{3} + 275\cdot 3\dot{4} + 1896\cdot\dot{5} + 73\cdot 035\dot{7} + \cdot\dot{2}$.

Ans. 2246·448.

Add $72\cdot\dot{3}\dot{4} + 281\cdot\dot{0}7\dot{6} + \cdot\dot{3}1\dot{5} + \cdot 02\dot{6} + 512\cdot 7\dot{4}\dot{8} + 176\cdot\dot{3}$.

Ans. $1042\cdot 84\dot{4}0021\dot{8}$.

Reduce to decimals and add $\frac{3}{11} + \frac{4}{7} + \frac{5}{8} + \frac{2}{13} + \frac{5}{9} + \frac{13}{27}$.

Ans. $2\cdot\dot{6}6003\dot{9}$.

Reduce to decimals and add $\frac{3}{14} + \frac{2}{5} + \frac{10}{11} + \frac{5}{13} + \frac{22}{21}$.

Ans. $2\cdot 95\dot{5}611\dot{0}$.

II. Subtract 39·987 from 74·872. Ans. 34·885.

Subtract 2·176 from 8·7214. Ans. 6·5454.

Subtract ·00819 from 1. Ans. ·99181.

Subtract $\cdot 0\dot{3}$ from 2·7146 Ans. $2\cdot 6812\dot{6}$.

Subtract $1\cdot 4\dot{6}$ from $3\cdot 714\dot{2}$. Ans. $2\cdot 247\dot{5}$.

Subtract $2\cdot\dot{7}1\dot{4}$ from $18\cdot\dot{1}76\dot{5}$. Ans. $15\cdot\dot{4}6180293705\dot{0}$.

Reduce and subtract $\frac{7}{11}$ from $2\frac{13}{14}$. Ans. $2\cdot 2\dot{9}2207\dot{7}$.

Reduce and subtract $7\frac{3}{13}$ from $87\frac{4}{9}$. Ans. $80\cdot\dot{2}1367\dot{5}$.

III. Multiply 2·175 by 3·18. Ans. 6·9165.

Multiply ·00194 by ·0357. Ans. ·000069258.

Multiply 1·721$\dot{3}$ by 876·5. Ans. 1508·748$\dot{6}$.

Multiply 1·7$\dot{1}$4$\dot{6}$ by 37·96. Ans. 65·086$\dot{7}$7$\dot{0}$.

Multiply 1·7$\dot{1}\dot{3}$6 by 2·5$\dot{3}$. Ans. 4·341$\dot{2}$0538$\dot{7}$.

Reduce and multiply $\frac{3}{11}$ by 2$\frac{4}{7}$. Ans. ·$\dot{7}$01298$\dot{7}$.

Reduce and multiply 2$\frac{5}{7}$ by 3$\frac{1}{3}$. Ans. 9·$\dot{0}$4761$\dot{9}$.

Reduce and multiply $\frac{2}{7}$ of 8$\frac{5}{9}$ by $\frac{1}{4}$ of 7$\frac{1}{11}$. Ans. 4·$\dot{3}$.

IV. Divide 75·314 by 275·6. Ans. ·27327286 —.

Divide 3·14862 by 875·723. Ans. ·00359545 +.

Divide 7.186$\dot{3}$ by 75·62. Ans. ·09503218 —.

Divide 91·$\dot{8}$6$\dot{3}$ by 87·5$\dot{6}$. Ans. 1·0490734 +.

Divide 30·$\dot{4}$1$\dot{6}$ by 21·$\dot{7}$3$\dot{2}$. Ans. 1·399567 +.

Reduce and divide 27$\frac{5}{9}$ by 3$\frac{7}{11}$. Ans. 7·5$\dot{7}$.

Reduce and divide 56$\frac{4}{7}$ by $\frac{1}{4}$ of 2$\frac{1}{9}$. Ans. 107·18797 —.

Reduce and divide $\frac{2}{7}$ of 56$\frac{1}{3}$ by $\frac{3}{8}$ of 7$\frac{5}{12}$. Ans. 5·787052 —.

V. Promiscuous Exercises in Vulgar and Decimal Fractions.

1. What number taken from $\frac{2}{3}$ of $\frac{3}{5}$ of 27$\frac{1}{2}$ will leave 10 for a remainder? Ans. 1.

2. There are three partners in a mercantile house whose shares are respectively $\frac{1}{3}$, $\frac{2}{7}$, and $\frac{8}{21}$, and they propose to admit a fourth partner whose share is to be $\frac{1}{5}$ of each of the three others. What will be the share of each after his admission?

Ans. $\frac{8}{27}$, $\frac{16}{63}$, $\frac{64}{189}$, and $\frac{21}{189}$.

3. A cistern has four pipes; by the first it can be filled in 12 hours, by the second in 11$\frac{1}{2}$ hours, by the third in 10 hours, and by the fourth in 8$\frac{1}{3}$ hours. In what time could it be filled by the four jointly, and also taking them two by two?

Ans. By the four jointly in 2$\frac{1514}{2693}$ hours; by the 1st and 2d in 5$\frac{41}{47}$ hours; 1st and 3d in 5$\frac{5}{11}$ hours; 1st and 4th in 4$\frac{56}{61}$ hours; 2d and 3d in 5$\frac{15}{43}$ hours; 2d and 4th in 4$\frac{99}{119}$ hours; and by the 3d and 4th in 4$\frac{6}{11}$ hours.

4. A gentleman left $\frac{2}{3}$ of $\frac{9}{10}$ of his estate to his eldest son, $\frac{3}{4}$ of $\frac{5}{8}$ of the remainder to his second son, $\frac{1}{2}$ of $\frac{2}{3}$ of what remained after this to his daughter, and £2000 to his widow. What was the share of each, and his whole property?

Ans. Eldest son, £8470$\frac{10}{17}$; second son, £2647$\frac{1}{17}$; the daughter, £1000. Whole sum left, £14,117$\frac{11}{17}$.

5. Three persons, A, B, and C, are joint proprietors of a ship; A, whose share was $\frac{3}{10}$, sold $\frac{1}{11}$ of it to B, whose share was originally

$\frac{4}{13}$, and he now disposes of $\frac{1}{6}$ of his augmented share to C for £700. What are now their respective shares and the value of the ship?

Ans. A's share $\frac{3}{11}$, B's $\frac{479}{1716}$, and C's $\frac{9997}{22308}$; and value of ship £12,538$\frac{298}{479}$.

6. What number is that which being multiplied by $\frac{2}{7}$ of $\frac{3}{4}$ of $9\frac{1}{10}$ produces 75? Ans. $38\frac{6}{13}$.

7. A captain sends out $\frac{1}{5}$ of his soldiers + 10, and he has remaining $\frac{1}{2}$ + 15. How many soldiers had he? Ans. 150.

8. A general being asked the number of men under his command, said, if you add $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of the number to itself and then subtract 5000, the remainder will be 100,000. What was the number?

Ans. 50,400.

9. A person left his second son £100, which was exactly $\frac{1}{4}$ of $\frac{2}{3}$ of his oldest son's portion, and $2\frac{1}{2}$ times the whole estate was equal to $7\frac{1}{3}$ times the oldest son's portion. What was the value of the estate? Ans. £1760.

10. Divide £4000 among A, B, and C; and give A $7\frac{1}{2}$ as often as B $5\frac{3}{4}$, and C $4\frac{3}{7}$ as often as B $6\frac{1}{3}$. What are their shares?

Ans. A £1824$\frac{12}{37}$, B £1398$\frac{24}{37}$, C £777$\frac{1}{37}$.

11. A, B, and C, can dig a trench in 6 days, A, B, and D, in 5 days, A, C, and D, in $5\frac{1}{2}$ days, and B, C, and D, in $4\frac{3}{4}$ days. In what time can they do it working together, and in what time can it be done by each working alone?

Ans. Jointly in $3\frac{4533}{4759}$ days; A $23\frac{433}{799}$, B $14\frac{64}{1339}$, C $18\frac{864}{997}$, and D $11\frac{946}{1624}$.

12. A cistern has 3 supply and 2 waste pipes; by the first and second it can be filled in 3 hours, by the first and third in $3\frac{1}{2}$ hours, by the second and third in 4 hours; the first waste-pipe empties it when full in 11 hours, and the second in 6 hours. In what time would it be filled by the three pipes jointly, and also by each separately, and in what time would it be filled if all the pipes were kept running?

Ans. Jointly in $2\frac{22}{73}$ hours, by the first in $5\frac{13}{31}$ hours, by the second in $6\frac{18}{25}$ hours, by the third in $9\frac{15}{17}$ hours, and when all are running in $5\frac{213}{327}$ hours.

13. A wine-merchant at one time drew from a cask of brandy $\frac{1}{3}$, at another time $\frac{1}{4}$ of the remainder, and at a third time $\frac{1}{5}$ of what was left, and he then found that there were only 10 gallons in the cask. How much was in it at first? Ans. 25 gallons.

14. A person being asked the hour of the day, replied that the day consisted of 16 hours, and the sun rises at 4 o'clock, and that if $\frac{1}{2}$ of the hours already past be added to $\frac{2}{3}$ of those remaining, the sum would be the number of hours from sunrise? What was the exact time? Ans. $8\frac{4}{7}$ minutes past 1 o'clock.

15. A, B, and C, are to share £100,000 in the proportion of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$ respectively; but C's part falling into the other two by his death, how is the whole sum to be divided between A and B?

Ans. A gets £57,142$\frac{282}{329}$, and B gets £42,857$\frac{47}{329}$.

16. A smuggler had a quantity of brandy, which he expected

would realize £9, 18s.; but after he had sold 10 gallons, a revenue-officer seized $\frac{1}{3}$ of the remainder, in consequence of which he realizes only £8, 2s. What was the price he sold it at per gallon, and the number of gallons he had at first?

Ans. 9s. per gallon, and the quantity at first 22 gallons.

17. If I have $\frac{5}{16}$ of a ship worth £36,000, and I dispose of $\frac{7}{8}$ of $\frac{5}{6}$ of $\frac{1}{11}$ of my share, what part have I left, and what is it worth?

Ans. $\frac{2465}{8448}$, value £10,504$\frac{69}{264}$.

18. A gentleman left to the elder of his two sons $\frac{40}{87}$ of his estate, and $\frac{30}{73}$ of the remainder to another; now it turned out that the share of the elder was just £15,000 more than that of the second; he left £10,000 to an only daughter, and the remainder to his widow. What was her share, and the value of the whole estate?

Ans. Widow's share £10,076$\frac{24}{151}$, and whole estate £63,089$\frac{61}{151}$.

19. A person went to market and paid away $\frac{2}{3}$ of the money he took with him; he then received payment of a bill of £45, and afterwards paid away $\frac{1}{2}$ of the money he then had; in going home he was robbed of £30, after which he had just 10s. left? How much money did he take away with him? Ans. £48.

20. Divide £1000 among 6 men, so that 4 may have each a share, the fifth man only $\frac{1}{4}$ of a share, and the sixth $\frac{1}{10}$ of a share.

Ans. Each of the four gets £229$\frac{77}{87}$, the fifth gets £57$\frac{41}{87}$, and the sixth gets £22$\frac{86}{87}$.

21. A person having $\frac{1}{3}$ of a lead mine, sold $\frac{3}{4}$ of $\frac{1}{10}$ of his share for £1000. What was the value of the whole mine at that rate, and also of his remaining share?

Ans. Whole mine £40,000, his remaining share £12,333$\frac{1}{3}$.

22. A shepherd at a fair sold $\frac{3}{4}$ of his sheep to one person, $\frac{2}{3}$ of $\frac{3}{4}$ of the remainder to another, and to a third he sold $\frac{3}{4}$ of what remained, and he then had 1 sheep left. How many had he at first? Ans. 32.

23. My fishing-rod consists of four parts; the root is $\frac{1}{4}$ of the whole + 3 inches, the second part is $\frac{1}{4}$ of the whole + 1 inch, the third part is $\frac{1}{4}$ of the whole — 1 inch, and the top is 3 feet 9 inches. What is its whole length? Ans. 16 feet.

24. Find the three least whole numbers of which $\frac{1}{3}$ of the first, $\frac{2}{5}$ of the second, and $\frac{3}{7}$ of the third are equal to each other.

Ans. 18, 15, and 14.

25. How many yards were in a piece of cloth of which, after $\frac{1}{3}$ + 16 yards was sold, there remained $\frac{1}{4}$ + 34 yards?

Ans. 120 yards.

26. If $\frac{3}{5}$ of $\frac{5}{7}$ of $\frac{7}{10}$ of a ship be worth $\frac{1}{5}$ of $\frac{10}{11}$ of $\frac{11}{13}$ of the cargo and freight, which are £17,840, and if the frieght be $\frac{1}{10}$ of $\frac{5}{8}$ of the value of the ship, what is the value of the ship, cargo, and freight? Ans. Ship, £11,150; freight, £696$\frac{7}{8}$; and cargo, £17,143$\frac{1}{8}$.

27. A common consisting of 10,000 acres is to be divided among A, B, and C, in the proportion of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$ respectively; but B being in want of money, disposes of his portion to the other two, in proportion to their shares, for £56,000. What quantity will each receive, and what sum will each pay to B?

Ans. A's share is 6079$\frac{2}{325}$ acres, and he pays £32,000; C's share is 3920$\frac{323}{325}$ acres, and he pays £24,000.

PRAXIS OF EVOLUTION.

1. Extract the square root of 75625. Ans. 275.
2. Extract the square root of 13719616. Ans. 3704.
3. Extract the square root of ·00005476. Ans. ·0074.
4. Extract the square root of 4·7089. Ans. 2·17.
5. Extract the square root of $\frac{11}{16}$. Ans. ·82915 +.
6. Extract the square root of $7\frac{5}{8}$. Ans. 2 76134 +.
7. Extract the cube root of 20796875. Ans. 275.
8. Extract the cube root of 50817457664. Ans. 3704.
9. Extract the cube root of 21·024576. Ans. 2·76.
10. Extract the cube root of 284. Ans. 6·573139.
11. Extract the cube root of $\frac{1}{5}$. Ans. ·5848035.
12. Extract the cube root of $3\frac{1}{8}$. Ans. 1·462.
13. Required the side of a square whose area is 7489 poles. Ans. 85·859 poles.

14. Required the side of a square whose area is equal to that of a rectangle whose length is 475 feet and breadth 40 feet. Ans. 137·8405.

15. Required the side of a cube whose area is 74896 yds. Ans. 42·152.

16. Find a mean proportional between 64 and 100. Ans. 80.

17. There are five numbers in geometrical proportion, of which the least is 2 and the greatest is 512. What are the others? Ans. 8, 32, and 128.

18. The reckoning at an inn came to £1, 16s., and each man had as many threepences to pay as there were men in the company. How many were there? Ans. 12 men.

19. There are four numbers in geometrical progression, the first is 9, and the last 576. What are the other two? Ans. 36 and 144.

20. Find five mean proportionals between 2 and 1458. Ans. 6, 18, 54, 162, and 486.

21. The solid content of a globe of 1 inch diameter is ·5236 inch. What is the content of another whose diameter is 20 inches? Ans. 16 119 cubic inches.

22. How many dies $\frac{1}{3}$ inch in the side can be cut out of a cubical piece of ivory 6 inches in the side? Ans. 944.

23. The national debt of Great Britain amounts to £840,408,440. Now, if a sovereign is an inch in diameter, and 20 of them make an inch in thickness, what would be the side of a cube of sovereigns equivalent to this debt? Ans. 347·6 inches.

ALGEBRA.

DEFINITIONS.

Algebra is a general method of computation and of investigation, in which quantities are represented by letters, and their relations pointed out by characters.

CHARACTERS EXPLAINED.

1. $+$ *plus*, is the sign of addition; as $a+b$ signifies the quantity represented by b added to that represented by a.

2. $-$ *minus*, is the sign of subtraction; as $a-b$ denotes the quantity b taken from the quantity a.

The sign $\backsim$ is employed to denote the difference between two quantities, when it is not known which is the greater; as $a \backsim b$ signifies the difference between a and b: Also $a \pm b$ signifies the sum or difference of a and b.

3. $\times$ *into*, is the sign of multiplication; as $a \times b$ represents the product of a by b, or of b by a. Instead of this sign we often use a point, or write the letters together as in one word: thus $a.b$ or ab signifies $a \times b$.

4. $\div$ *by*, is the sign of division; but it is generally expressed by placing the dividend above the line and the divisor below it, in the form of a fraction: thus $a \div b$ or $\frac{a}{b}$ signifies a divided by b.

5. $: :: :$ is the sign of proportion; as $a:b::c:d$ is read, As a is to b, so is c to d.

6. $=$ *equal to*, is the sign of equality: thus $a=b$ signifies a is equal to b.

7. $> \; <$ are signs of greater and less: thus $a > b$, a is greater than b; $a < b$, a is less than b; the opening of the sign being always turned towards the greater quantity, and its angular point towards the less.

8. $7a$. A number prefixed to a letter is called its *coefficient*, and shews how often the letter is to be taken; as here, 7 times a. When no coefficient is expressed, the coefficient 1 is always understood: thus a and $1a$ denote the same thing.

9. $(a+b) \times c$ or $\overline{a+b} \times c$. A parenthesis enclosing letters, or a line drawn over them, is called a *vinculum*, and

points out how many are to be multiplied, divided, &c.; as here, the sum of a and b is to be multiplied by c.

10. aaa. When the same letter is repeated twice, or oftener, it is understood to be multiplied as often into itself, and the product is called a power of the quantity represented by that letter: thus aa is the second power or square of a, aaa is the third power of a, &c.; and in relation to these powers the quantity is called the first power of itself.

11. a^3. Instead of repeating the same letter, we generally place a figure above it towards the right hand, to shew how often it is repeated; as a^3 is the third power of a, a^4 the fourth power, a^n the power of a denominated by the number n.

12. The character placed above is called the exponent or index of the power.

13. The *root* of any quantity or power is a quantity which, if multiplied by itself a certain number of times, produces the original quantity or power; and is denoted by *the radical sign* $\sqrt{\ }$: thus $\sqrt{9}$ is the square root of 9, $\sqrt[3]{8}$ is the cube root of 8, $\sqrt[4]{81}$ is the fourth root of 81.

14. A fractional exponent or index is more generally used to express the root, and then the upper figure denotes the power, and the under figure the root: thus $a^{\frac{2}{3}}$ is the third root of the second power of a, $a^{\frac{1}{4}}$ is the fourth root of the first power of a, or of a itself.

15. A *simple* quantity is that which consists of but one term; as a, ab, $4abc$, &c.

16. A *compound* quantity consists of two or more simple terms, connected by the signs $+$ or $-$; as $a+b$, $4a-3b+6ac$, &c. If a compound quantity consists of only two terms, it is called a *binomial;* if of three, a *trinomial;* if of four, a *quadrinomial;* and if it consists of more than four, a *polynomial* or *multinomial.*

17. *Like* terms are those of which the literal parts are the same, *i. e.* consist of the same letters; as $4ab$, ab, $9ab$, &c.

18. *Unlike* terms are those which consist of different letters; as $2ab$, $3bc$, $5cd$, &c.

19. The sign $\therefore$ is sometimes used to denote the words *therefore* or *consequently.*

Quantities which have the sign $+$ before them are said to be positive or *additive,* and those which have the sign $-$ negative or *subtractive.* A quantity which has no sign prefixed is understood to have $+$.

The following examples will illustrate these characters, and shew their use, in which any values may be affixed to the letters:—

Let $a=12 \quad c=2 \quad e=5 \quad g=25 \quad i=11$
$\quad b=3 \quad d=4 \quad f=9 \quad h=7 \quad k=1.$

1. $a+b-c+d$ . . . $=$ 17.
2. $4a-5b+4c-7d$ . . . $=$ 13.
3. $ab-2cd+4be-3cf$ . . $=$ 26.
4. $8a^2-5ab+10ac-4bc+4b^2$ . . $=$ 1224.
5. $6a^3-4a^2b+2ab^2-7b^3$ . . $=$ 8667.
6. $2a^2bc+3ab^2c-5abc^2$ . . $=$ 1656.
7. $(a-b)\times 2c-d\times(b+c)$ . . $=$ 16.
8. $(a+b)\times(g-h)\times(i-k)$ . . $=$ 2700.
9. $2a^2c-\frac{a^2}{c}+\frac{a}{c^2}$ $=$ 507.
10. $\frac{3a}{c}+\frac{4g}{5d}-\frac{6d}{b}$ $=$ 15.
11. $\frac{a+b+e}{d+k}+\frac{g}{e}$ $=$ 9.
12. $\frac{3a}{b}\times\frac{2c}{d}\times\frac{4g}{d}$ $=$ 300.
13. $\frac{ac}{d}+\frac{gf}{c}-\overline{cd}|^{\frac{1}{3}}$ $=$ $116\frac{1}{2}$.
14. $\frac{\sqrt{ac+d}}{i-k}$. $=$ 2.
15. $(g+c)^{\frac{1}{3}}+(ai-b^2-c)^{\frac{1}{2}}$ . . . $=$ 14.

NOTE. All the fundamental operations of algebra depend upon this single principle, viz. When a quantity is to be increased or diminished by other quantities, the same result will be obtained in whatever order the procedure is carried on, provided none of the quantities be neglected. This is manifest from the nature of quantity, which has no relation to order. Thus, if we have to add 7 and 5, and to subtract 3, we may first subtract 3 from 7, and add the remainder to 5; or we may subtract 3 from 5, and add the remainder to 7; or we may add 7 to 5, and from the sum 12 subtract 3: the result in every case is 9. Again, if we have to multiply 12 and 6, and to divide by 3; we may first divide 12 by 3, and multiply the quotient by 6; or we may divide 6 by 3, and multiply the quotient by 12; or we may multiply 12 by 6, and divide the product by 3: the result in every case is 24.

ADDITION.

CASE 1. WHEN the quantities are alike; if the signs be the same, add the coefficients, but if not the same, take their difference, and to the sum or difference prefix the sign of the greater, and annex the common letter or letters.

CASE 2. When the quantities are unlike; write them one after another, with their proper signs and coefficients.

NOTE. When there are more than two like quantities, add the coefficients of those which have + into one sum, and of those which have — into another, and subtract the less sum from the greater. The arrangement of the quantities is arbitrary, and must often be altered to bring like quantities under like.

1. $3a-5b+4c-3d-2e$
$\underline{6a+2b-7c-4d+8e}$
$9a-3b-3c-7d+6e.$

2. $8a^2b-5ab^2-8abc+4bc^2$
$-2a^2b+6ab^2-\ abc-4bc^2.$

3. $6ab+2ac-3bc+4bd$
$-7ab-3ac+6bc+5bd.$

4. $8a^{\frac{1}{2}}b^3-7a^2bc^{\frac{1}{2}}-4ab^{\frac{1}{2}}c^2$
$7a^{\frac{1}{2}}b^3+7a^2bc^{\frac{1}{2}}-3ab^{\frac{1}{2}}c^2.$

5. $8a^3b-7a^2b^2+4ab^3-a^4$
$7a^2b^2-8ab^3+4a^4-2a^3b$
$6ab^3-2a^4-7a^3b+5a^2b^2$
$5a^4-6a^3b+5a^2b^2-3ab^3$
$2a^2b^2-2a^3b-ab^3+4a^4.$

6. $a+(a-v)^{\frac{1}{2}}+\ 5$
$2a+(a-v)^{\frac{1}{2}}-10.$

7. $a+(a+v)^{\frac{1}{2}}+\ 5$
$2a+(a-v)^{\frac{1}{2}}-10.$

8. a^3+a^2-a
$a^{\frac{5}{2}}+a^{\frac{2}{3}}-a^{\frac{1}{2}}$
$a^{\frac{5}{2}}+a^2-a^{\frac{1}{3}}.$

9. $10(a+e)^{\frac{1}{2}}+(a-e)^{\frac{1}{2}}$
$-(a+e)^{\frac{1}{2}}-(a-e)^{\frac{1}{2}}.$

10. $a^3+3a^2+5+(a-v)^{\frac{1}{2}}+a+6(a+v)^{\frac{1}{2}}$
$3a^2-2a+6a^3-2(a-v)^{\frac{1}{2}}+10-6(a+v)^{\frac{1}{2}}$
$7a-5a^3-2a^2+4(a+v)^{\frac{1}{2}}-b+8(a-v)^{\frac{1}{2}}$
$8c-6a^2+4a^3-2(a-v)^{\frac{1}{2}}+7-6a$
$7a^2-8a^3+4-5(a+v)^{\frac{1}{2}}+3a-8(a+v)^{\frac{1}{2}}.$

NOTE. If the difference $a-b$ is to be added to $3a$, we may first subtract b from a, and then add the remainder to $3a$; or we may subtract b from $3a$, and add a to the remainder. Here we first add a to $3a$, and then subtract b, and it becomes $4a-b$. If $2a+b$ is to be added to $3a-4b$, we add $2a+b$ to $3a$, and it becomes $5a+b$; from which we take $4b$, and it becomes $5a-3b$.

SUBTRACTION.

CHANGE the signs of the subtrahend from $+$ to $-$, or from $-$ to $+$, and then proceed as in Addition.

1. $8ab-2cd+5ac-7ad$
$\underline{3ab+4cd+5ac-2ad}$
$5ab-6cd\quad * \quad -5ad.$

2. $18a^2b-12abc-3ab^2+b^3$
$6a^2b+\ 3abc-4ab^2-3b^3.$

3. $a^2x^2c-5ax^2c^2+2a^2xc^2$
$3a^2x^2c+4ax^2c^2+2a^2xc^2.$

4. $-3a^3b^{\frac{1}{2}}+2a^2bc^{\frac{1}{2}}-5a^{\frac{1}{2}}b^2c$
$4a^3b^{\frac{1}{2}}-2a^2bc^{\frac{1}{2}}-5a^{\frac{1}{2}}b^2c.$

5. $3bd+2a$
$2bd-3a-b.$

6. $\dfrac{(a-b+2)^{\frac{1}{2}}}{a+b}$
$-\left(\dfrac{a-b+2}{a+b}\right)^{\frac{1}{2}}.$

7. $2bc-11a-d$
$d+11a-2bc.$

8. $a^3 + a^{\frac{3}{2}}$
$a^3 - a^{\frac{3}{2}}$.

9. $6a^{\frac{1}{2}} - 4b^{\frac{2}{3}} + x^2$
$4x^2 - 3a^{\frac{1}{2}} + 2b^{\frac{2}{3}}$.

NOTE. If we are to subtract $a - c$ from $3a$, we may first subtract c from a, and then subtract the remainder from $3a$; or we may add c to $3a$, and then subtract a from the sum. Here we subtract the whole a from $3a$, and add c to the remainder. If $a - c$ is to be subtracted from $3a + 2c$, we subtract a as before from $3a$, and then add c, and the remainder becomes $2a + 3c$. Now all this is performed by changing the signs of the quantity $a - c$ into $-a + c$, and then adding it.

These considerations lead us to perceive how we may add or subtract any two terms, without regard to the other terms with which they are connected.

MULTIPLICATION.

MULTIPLY the coefficients, and to the product annex the letters of both factors.

If the sign of the multiplier is $+$, make the sign of the product the same with that of the multiplicand. If the sign of the multiplier is $-$, make the sign of the product contrary to that of the multiplicand.

Hence, like signs produce $+$, and unlike signs $-$.

If the multiplicand is compound, multiply each term of it separately by the multiplier.

If the multiplier is compound, multiply first by one of its terms, then by another, &c. and afterwards add the products.

Powers of the same quantity are multiplied by adding their exponents.

1. Multiply $5a - 4b + 3c - 2d + e - 1$
by $5a$

$$25a^2 - 20ab + 15ac - 10ad + 5ae - 5a.$$

2. Multiply $6a^2 - 7ab + 4ac - b^2 + 2bc - c^2$ by $4ab$.*
3. . . . . . . $3a - 2b$ by $-2a + 4b$.
4. . . . . . . $5a^2 - 3ab + 4b^2$ by $6a - 5b$.
5. . . . . . . $a^2 + ab + b^2$ by $a - b$.
6. . . . . . . $a^4 - x^4$ by $a^4 - x^4$.
7. . . . . . . $2x^2 - 3xy + 6$ by $3x^2 + 3xy - 5$.
8. . . . . . . $5a^2 - 4ax + 3x^2$ by $2a^2 - 3ax - 4x^2$.

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* ANSWERS. (2.) $24a^3b - 28a^2b^2 + 16a^2bc - 4ab^3 + 8ab^2c - 4abc^2$. (3.) $-6a^2 + 16ab - 8b^2$. (4.) $30a^3 - 43a^2b + 39ab^2 - 20b^3$. (5.) $a^3 - b^3$. (6.) $a^8 - 2a^4x^4 + x^8$. (7.) $6x^4 - 3x^3y + 8x^2 - 9x^2y^2 + 33xy - 20$. (8.) $10a^4 - 23a^3x - 2a^2x^2 + 7ax^3 - 12x^4$.

9. Multiply $2a^2x^2-2ax+3a^2$ by $3a^2x^2+4ax-5a^2$.*

10. ...... $x^2-ax+\frac{1}{4}a^2$ by $x^2+ax-\frac{1}{4}a^2$.

11. ...... $x-\frac{1}{2}a$ by $x+\frac{1}{2}a$.

12. ...... x^2+xy+y^2 by x^2-xy+y^2.

13. ...... $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$.

14. ...... $3a-2b+2c$ by $2a-4b+5c$.

15. ...... $a^3-3a^2b+3ab^2-b^3$ by $a^2-2ab+b^2$.

16. ...... a^3-3a^2+3a-1 by a^2-2a+1.

NOTE. Since $1\times b+1\times b+1\times b=3\times b$, if as many units be taken as are in a, and each of them be multiplied by b and the products be added, the sum will be $a\times b$; but b taken as many times as there are units in a produces $b\times a$; therefore $a\times b$ is the same with $b\times a$, or $ab=ba$. In like manner abc, acb, bac, bca, cab, cba, are all the same, so that the factors may be placed in any order.

Again, since $ma=a+a+a$, &c. being repeated m times,
and $mb=b+b+b$, &c. being repeated m times;
therefore $ma+mb=(a+b)+(a+b)+(a+b)$ repeated m times, that is, $ma+mb=m(a+b)$. In like manner $ma-mb=m(a-b)$.

In multiplying $a-b$ by c, we may either first subtract and then multiply, or first multiply and then subtract. The latter is the order in algebra: we first multiply a by c, which makes ac, and then b by c, which makes bc, and subtract the latter product from the former to get the just product $ac-bc$, where the signs are the same with those of the multiplicand.

In multiplying $a-b$ by $c-d$, we first multiply $a-b$ by c as before, and it produces $ac-bc$; then we multiply $a-b$ by d, and it produces $ad-bd$, which we subtract from the former product, or change its signs, and it becomes $-ad+bd$, where the signs are contrary to those of the multiplicand.

The first and last terms shew that quantities with like signs produce +, and the other two terms shew that those which have unlike signs produce —.

DIVISION.

WHEN the divisor is a simple quantity, write it under the dividend in the form of a fraction, then cancel like quantities in them, and divide the coefficients by their greatest common measure.

When the signs are alike, the sign of the quotient is + ; but if they be unlike, it is —.†

Powers of the same quantity are divided by subtracting the

* ANSWERS. (9.) $6a^4x^4+2a^3x^3-a^4x^2-8a^2x^2+22a^3x-15a^4$. (10.) $x^4-a^2x^2+\frac{1}{2}a^3x-\frac{1}{16}a^4$. (11.) $x^2-\frac{1}{4}a^2$. (12.) $x^4+x^2y^2+y^4$. (13.) $10a^4-27a^3x+34a^2x^2-18ax^3-8x^4$. (14.) $6a^2-16ab+19ac+8b^2-18bc+10c^2$. (15.) $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$. (16.) $a^5-5a^4+10a^3-10a^2+5a-1$.

† This is evident; for the divisor multiplied by the quotient must produce the dividend with its proper sign. The whole operation depends upon this principle, that the value of a quantity is not altered by both multiplying and dividing it by the same quantity.

exponent of the divisor from that of the dividend; the remainder is the exponent of the quotient.

If the dividend be compound, divide each term of it separately by the divisor.

Divide the following:

1. $56a^2b^3c$ by $8ab^3$. . . . Ans. $7ac$.

2. $54xy^2$ by $36x^2y$. . . . $\frac{3y}{2x}$.

3. $63a^3b^2c^3-42a^2b^3c^3$ by $14a^2b^2c^2$. . $\frac{9ac}{2}-3bc$.

4. $24x^3y-18x^2y^2+15xy^3$ by $30xy^2$. . $\frac{4x^2}{5y}-\frac{3x}{5}+\frac{y}{2}$.

When the divisor is compound, arrange the terms of the dividend and divisor according to the powers of the same letter. Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient, then multiply the whole divisor by this term, and subtract the product from the dividend; bring down as many terms to the remainder as is requisite for a new dividual, with which proceed as before.

NOTE. When the last remainder is a simple quantity, place the divisor below it in the form of a fraction, and annex it with its proper sign to the quotient.

5. Divide $a^3-3a^2b+3ab^2-b^3$ by $a-b$.

$$
\begin{array}{l}
a-b)a^3-3a^2b+3ab^2-b^3(a^2-2ab+b^2.\\
\quad\;\; \underline{a^3-\;\;a^2b}\\
\qquad\quad -2a^2b+3ab^2\\
\qquad\quad \underline{-2a^2b+2ab^2}\\
\qquad\qquad\qquad +ab^2-b^3\\
\qquad\qquad\qquad +ab^2-b^3.
\end{array}
$$

6. $8a^3-4a^2b-6ab^2+3b^3$ by $2a-b$. Ans. $4a^2-3b^2$.

7. $3b^3+3ab^2-4a^2b-4a^3$ by $a+b$. . $-4a^2+3b^2$.

8. a^4-b^4 by $a-b$. . . $a^3+a^2b+ab^2+b^3$.

9. $8a^4+2a^2b^2-3b^4$ by $2a^2-b^2$. . $4a^2+3b^2$.

10. $2a^2x^2-5ax+2$ by $2ax-1$. . $ax-2$.

11. $x^2-x+\frac{1}{4}$ by $x-\frac{1}{2}$. . . $x-\frac{1}{2}$.

12. $21a^5-21b^5$ by $7a-7b$.
Ans. $3a^4+3a^3b+3a^2b^2+3ab^3+3b^4$.

13. $x^4-y^4+2y^2z^2-z^4$ by $x^2+y^2-z^2$. . $x^2-y^2+z^2$.

14. $1+a$ by $1-a$. . $1+2a+2a^2+2a^3+2a^4+$, &c.

15. $8x^2-15y^2+23yz-2xy-8xz-6z^2$ by $2x-3y+z$. Ans. $4x+5y-6z$.

16. $a^2-2ab+b^2$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$. . $a^{\frac{3}{2}}-ab^{\frac{1}{2}}-a^{\frac{1}{2}}b+b^{\frac{3}{2}}$.

17. $6x^4-96$ by $3x-6$. . . $2x^3+4x^2+8x+16$.

18. $1+2x$ by $1-x$. . . $1+3x+3x^2+3x^3+$, &c.

FRACTIONS.

A FRACTION is one or more parts of a unit. The denominator expresses the number of parts into which the unit is supposed to be divided, and the numerator expresses the number of these parts of which the fraction consists: thus, in the fraction $\frac{m}{n}$, n denotes the number of parts into which the unit is divided, and m points out the number of these parts of which the fraction consists. If the unit had been divided into $2n$ parts, then the fraction must have consisted of twice the number of these parts, and would have been $\frac{2m}{2n}$. In the same manner it might be expressed by $\frac{3m}{3n}$, $\frac{rm}{rn}$, &c.

Hence, the value of a fraction is not altered by multiplying or dividing both its terms by the same quantity.

REDUCTION.

PROBLEM I.

To reduce an integer or a mixed quantity to the form of a fraction.

If the denominator be given, multiply the integer by it for the numerator, and under the product place the denominator. If no denominator is given, place unit for it.

Hence, a mixed quantity may be reduced to the form of a fraction by multiplying the integer by the denominator of the fraction, and adding the numerator to the product for the numerator, below which place the denominator.

1. Reduce $3a$ to a fraction, of which the denominator is $2b$.

Ans. $\frac{6ab}{2b}$.

2. Reduce $a+\frac{b}{c}$ to an improper fraction. . $\frac{ac+b}{c}$.

3. . . . . $x+\frac{a^2}{x}$. . . . $\frac{x^2+a^2}{x}$.

4. Reduce $x-\frac{a^2x^2}{x}$ to an imp. frac. Ans. $\frac{x^2-a^2x^2}{x}$.

5. . . . . $5-\frac{3x}{a}$. . . . $\frac{5a-3x}{a}$.

6. . . . . $a-\frac{ab-a^2}{2b}$.* . . $\frac{ab+a^2}{2b}$.

7. . . . . $a-x-\frac{a^2x^2}{2x}$. . . $\frac{2ax-2x^2-a^2x^2}{2x}$

8. . . . . $a+1-\frac{x-1}{b}$. . . $\frac{ab+b-x+1}{b}$.

9. . . . . $1+3a-\frac{4x-5}{4x}$. . . $\frac{12ax+5}{4x}$.

PROBLEM II.

To reduce an improper fraction to an integer or a mixed quantity.

Divide the numerator by the denominator, the quotient is the integer, to which annex the remaining terms, with their proper signs, and the result will be the mixed number required.

1. Reduce $\frac{ab+b^2}{a}$ to a mixed quantity. Ans. $b+\frac{b^2}{a}$.

2. . . . . $\frac{ax+2x^2}{a+x}$. . . . $x+\frac{x^2}{a+x}$.

3. . . . . $\frac{x^2-y^2}{x+y}$. . . $x-y$.

4. . . . . $\frac{x^3-y^3}{x-y}$. . . . x^2+xy+y^2.

5. . . . . $\frac{12x^2-18}{3x}$. . . $4x-\frac{6}{x}$.

6. . . . . $\frac{4x^2-2x}{2x^2-x+1}$. . . $2-\frac{2}{2x^2-x+1}$.

PROBLEM III.

To reduce fractions of different denominators to others of the same value which have a common denominator.

Multiply each of the numerators into all the denominators, except its own, for the new numerators, and all the denominators together for the common denominator.

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* When a fraction has the sign — before it, all the signs of the numerator are to be changed. Here $ab-a^2$ becomes $-ab+a^2$.

1. Reduce $\frac{3a}{b}$ and $\frac{2a}{3c}$ to a common denominator.

Ans. $\frac{9ac}{3bc}$ and $\frac{2ab}{3bc}$.

2. . . . . $\frac{1}{a+b}$ and $\frac{1}{a-b}$. . . $\frac{a-b}{a^2-b^2}$ and $\frac{a+b}{a^2-b^2}$.

3. . . . . $\frac{a}{1-x}$ and $\frac{b}{1+x}$. . . $\frac{a+ax}{1-x^2}$ and $\frac{b-bx}{1-x^2}$.

4. . . . . $\frac{x-y}{x+y}$ and $\frac{x+y}{x-y}$. $\frac{x^2-2xy+y^2}{x^2-y^2}$ and $\frac{x^2+2xy+y^2}{x^2-y^2}$.

5. . . . . $\frac{a+b}{c}$ and $\frac{3d}{m}$. . . . $\frac{am+bm}{cm}$ and $\frac{3cd}{cm}$.

6. . . . . $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{m}{n}$. . . . $\frac{adn}{bdn}$, $\frac{bcn}{bdn}$, $\frac{bdm}{bdn}$.

7. . . . . $\frac{2a}{3}$, $\frac{3b}{4}$, and $\frac{5c}{3d}$. . . $\frac{8ad}{12d}$, $\frac{9bd}{12d}$, $\frac{20c}{12d}$.

8. . . . . $2a$ and $\frac{3b}{4}$. . . . $\frac{8a}{4}$ and $\frac{3b}{4}$.

9. $\frac{7a^2}{x}$, $\frac{a}{4}$, $\frac{a^2-x^2}{a+x}$. . $\frac{28a^3+28a^2x}{4ax+4x^2}$, $\frac{a^2x+ax^2}{4ax+4x^2}$, $\frac{4a^2x-4x^3}{4ax+4x^2}$.

PROBLEM IV.

To reduce a fraction to lower terms.

Divide its numerator and denominator by any quantity which measures both.

The greatest divisor of the coefficients is found as in arithmetic, and the greatest simple divisor of the letters is discovered by inspection.

1. Reduce $\frac{ax^2-x^3}{ax+x^2}$ to lower terms. . Ans. $\frac{ax-x^2}{a+x}$.

2. . . . . $\frac{6a^2-12x^2}{3a-6x}$. . . . . $\frac{2a^2-4x^2}{a-2x}$.

3. . . . . $\frac{4a^2x^3}{2ax-2a^2}$. . . . . $\frac{2ax^3}{x-a}$.

4. . . . . $\frac{36a^2x^2}{24a^3x}$. . . . . $\frac{3x}{2a}$.

5. . . . . $\frac{9a^2-12ax+4x^2}{3ax-2x^2}$. . . $\frac{3a-2x}{x}$.

To find the greatest compound divisor.

Divide that term which is of the higher dimensions by the other and the divisor by the remainder continually, till nothing remains: the last divisor is the greatest common measure.

NOTE. The several divisors must be first divided by the greatest simple quantity which measures all their terms; and when the first term of a divisor is not contained an exact number of times in the first term of the dividend, the latter must be multiplied by any simple quantity that will make the division succeed. Also any compound quantity in a remainder which does not measure the divisor from which it proceeds, may be taken out of it. And when any of the divisors become negative, they may have all their signs changed without affecting the truth of the result.

What is the greatest common measure of

1. $\frac{a^4-b^4}{a^5+a^3b^2}$? . . . Ans. a^2+b^2.

2. $\frac{x^2-y^2}{x^4-y^4}$? . . . . x^2-y^2.

3. $\frac{x^4-y^4}{x^3-x^2y-xy^2+y^3}$? . . . x^2-y^2.

4. $\frac{6x^3-6x^2y+2xy^2-2y^3}{12x^2-15xy+3y^2}$? . . $x-y$.

5. $\frac{3bcq+30mp+18bc+5mpq}{24ad-7fgq-42fg+4adq}$?* . . $q+6$.

6. $\frac{x^3+ax^2+bx^2-2a^2x+bax-2ba^2}{x^2-bx+2ax-2ab}$? . $x+2a$.

7. Reduce $\frac{x^2-1}{xy+y}$ to its lowest terms. . $\frac{x-1}{y}$.

8. . . . . $\frac{ax+x^2}{ac^2+c^2x}$. . . Divide by $a+x$.

9. . . . . $\frac{x^3-a^2x}{x^2+2ax+a^2}$. . . . by $x+a$.

10. . . . . $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$. . . by a^2-x^2.

11. . . . . $\frac{5a^5+10a^4x+5a^3x^2}{a^3x+2a^2x^2+2ax^3+x^4}$. . by $a+x$.

12. . . . . $\frac{a^5+a^3b^2}{a^4-b^4}$. . . . by a^2+b^2.

ADDITION AND SUBTRACTION.

REDUCE the fractions to a common denominator, if they have different ones; then add or subtract their numerators, and

* In fractions like this, where a letter is but of one dimension in either the numerator or the denominator, divide it into two parts, one of which has that letter in every term; then find the common measure of these two parts, and try whether it will divide the other quantity. Here the parts of the denominator are $4adq+24ad$ and $-7fgq-42fg$, and the common measure of these is $q+6$, which succeeds.

under the sum or the remainder write the common denominator, for the sum or the difference of the fractions.

1. Add $\frac{3a}{4}$, $\frac{5a}{6}$, $\frac{a}{3}$ together. . Ans. $\frac{23a}{12}$.

2. . . $\frac{x-3}{4}$, $\frac{5x+2}{3}$, $\frac{7x}{5}$. . . . . $\frac{199x-5}{60}$.

3. . . $4x$, $\frac{3x^2}{2a}$, $\frac{x+a}{3x}$. . $\frac{9x^3+24ax^2+2ax+2a^2}{6ax}$.

4. . . $2a+\frac{a+3}{5}$, $4a+\frac{2a-5}{4}$. . $6a+\frac{14a-13}{20}$.

5. . . $\frac{x}{a}-\frac{x}{2a}$, $\frac{3x}{a}-\frac{4x}{2a}$, $4x$. . $4x+\frac{3x}{2a}$.

6. . . $x-\frac{a^2}{x}$, $a-\frac{a-x}{c}$. . $a+x+\frac{x^2-ax-a^2c}{cx}$.

7. From $3a-\frac{4x}{a}$, take $a+\frac{5x}{3a}$. . $2a-\frac{17x}{3a}$.

8. . . . $\frac{7x}{v}-\frac{4x^2}{5v}$, take $\frac{3x}{7v}-\frac{2x^2}{17v}$. . $\frac{46x}{7v}-\frac{58x^2}{85v}$.

9. . . . $\frac{x-y}{2a}$, take $\frac{x+y}{3a}$. . . . . $\frac{x-5y}{6a}$.

What is the sum and the difference of

10. $\frac{x+y}{2}$ and $\frac{x-y}{2}$? . . Sum x, Diff. y.

11. $\frac{1}{a-b}$ and $\frac{1}{a+b}$? . Sum $\frac{2a}{a^2-b^2}$, Diff. $\frac{2b}{a^2-b^2}$.

12. $2x+\frac{3x}{a}$ and $x-\frac{2x-2a}{3c}$?

Sum $3x+\frac{9cx-2ax+2a^2}{3ac}$, Diff. $x+\frac{9cx+2ax-2a^2}{3ac}$.

MULTIPLICATION AND DIVISION.

Multiply the numerators together for the numerator of the product, and the denominators together for its denominator.

In division, invert the divisor and work as in multiplication.

1. Multiply $\frac{2x}{3}$ by $\frac{5x}{6}$. . . Ans. $\frac{5x^2}{9}$.

2. . . . . $\frac{x+a}{a+c}$ by $\frac{a}{x}$. . . . $\frac{ax+a^2}{ax+cx}$.

3. . . . . $b+\frac{bx}{a}$ by $\frac{a}{x}$. . . $\frac{ab}{x}+b$.

4. . . . . $\frac{ad}{2bc}$ by $\frac{4c}{d}$. . . . $\frac{2a}{b}$.

5. Divide $\frac{x}{3}$ by $\frac{2x}{9}$. . . . Ans. $1\frac{1}{2}$.

6. . . . . $\frac{2x^2}{a^3+x^3}$ by $\frac{x}{x+a}$. . . $\frac{2x(x+a)}{a^3+x^3}$.

7. . . . . $\frac{x}{x-1}$ by $\frac{x}{2}$. . . . $\frac{2}{x-1}$.

8. . . . . $\frac{x^4-a^4}{x^2-2ax+a^2}$ by $\frac{x^2+ax}{x-a}$. . $\frac{x^2+a^2}{x}$.

9. . . . . $\frac{a+x}{b^2+2bx+x^2}$ by $\frac{1}{b+x}$. . . $\frac{a+x}{b+x}$.

NOTE. The four fundamental rules require the aid of those for fractions, when any terms of the given quantities, or of those which arise in the course of the operation, are fractional.

10. Multiply $\frac{a^2}{9}-\frac{ax}{3}+\frac{x^2}{4}$ by $\frac{a}{3}-\frac{x}{2}$. . Ans. $\left(\frac{a}{3}-\frac{x}{2}\right)^3$.

11. . . . . $\frac{a}{b}+\frac{c}{d}$ by $\frac{a}{b}-\frac{c}{d}$. . $\frac{a^2}{b^2}-\frac{c^2}{d^2}$.

12. . . . . $\frac{3a}{4b}+\frac{2c}{3d}$ by $\frac{3a}{4b}-\frac{2c}{3a}$. . . $\frac{9a^2}{16b^2}-\frac{4c^2}{9d^2}$.

13. . . . . $\frac{x^2}{a^2}+\frac{xy}{ac}+\frac{y^2}{c^2}$ by $\frac{x}{a}-\frac{y}{c}$. . $\frac{x^3}{a^3}-\frac{y^3}{c^3}$.

14. Divide a^2+b^2 by $a+b$. . . $a-b+\frac{2b^2}{a+b}$.

15. . . . . $\frac{x^2}{16}-\frac{xy}{6}+\frac{y^2}{9}$ by $\frac{x}{4}-\frac{y}{3}$. . $\frac{x}{4}-\frac{y}{3}$.

16. . . . . $\frac{x^3}{a^3}-\frac{z^3}{c^3}$ by $\frac{x}{a}-\frac{z}{c}$. . . $\frac{x^2}{a^2}+\frac{xz}{ac}+\frac{z^2}{c^2}$.

OF NEGATIVE QUANTITIES.

IF c be the difference between a and b, the algebraical expression for this is $a-b=c$, where a is supposed to be greater than b; if it be less, the expression is $a-b=-c$. As, however, a greater quantity cannot be taken from a less, the expression $-c$ is impossible; so that a negative quantity standing by itself has, strictly speaking, no meaning. But if it be joined to another quantity, as $m-c$, the expression is proper, and may be subjected to all the operations of algebra. The absurdity appears only in the result; and when it does appear, it points out that something impossible has been ad-

mitted into the question, some condition inconsistent with its other conditions. We therefore reckon a negative result to be a proper algebraical solution of a problem; for it agrees with the preceding steps of the process, and points out the impossibility of the conditions, and thus it has its use in limiting the terms of the question. It will therefore be necessary in what follows to attend to negative expressions, and the forms which result from them, as well as from the positive ones. But this should create no hesitation in the operations; for it has been shown, not only how whole quantities, but also how single terms of them, may be added together or subtracted from one another, and how they may be multiplied or divided by one another with the signs of the resulting terms. But it is to be remarked, that these signs do not belong to the terms taken as isolated quantities, but to the relation in which they stand to the other terms of the result. When Diophantus of old said, "A defect drawn into a defect produces an excess," he did not by *a defect* mean a simple quantity, without relation to any other quantity: he meant to express by it, what one quantity wanted to make it equal to another, and that after the sum of the products of the wholes by these defects had been subtracted from the product of the wholes, the true product would exceed the remainder by the product of the defects, which must therefore be added to the remainder. And that this is the case, has been proved before, in the note explaining Multiplication. It is therefore improper to apply to simple quantities the rules by which the terms of compound quantities are connected together; and much of the obscurity of algebra has arisen from this confusion.

If $a - x$ be multiplied by itself, the product is $a^2 - 2ax + x^2$; and if $x - a$ be multiplied by itself, the product is the same; so that from this product it cannot be determined whether a be greater or less than x; that is, if $a - x = c$, whether the product has arisen from $+c$ or from $-c$, for each of these multiplied by itself produces $+c^2$, and therefore the square root of $+c^2$ may be either $+c$ or $-c$, and of course the square root of $-c^2$ is impossible. This expression is in some instances found useful for promoting the investigation of rules.

The formula $a^2 - b^2 = (a+b) \times (a-b)$ is useful in every branch of the mathematics. Now $a^2 + b^2 = a^2 - b^2 \times -1 = (a + b\sqrt{-1}) \times (a - b\sqrt{-1})$. This latter expression is therefore useful in several investigations.

The algebraist does not consider the solution of a problem to be complete, unless it exhibit all the cases which can occur; and the results which flow from contradictory suppositions can

only be exhibited by such expressions as have been just now explained.

In the application of algebra to various sciences, where position and other states must be introduced, quantities are often found in such opposite states, that when in one of them they are to be added, they must be invariably subtracted in the other. These different states may therefore be naturally pointed out by prefixing the sign $+$ to the quantity when it is in one of them, and the sign $-$ when it is in the opposite state; and this use does not appear to alter in the smallest degree the meaning affixed to these signs in the definitions, for here they are prefixed solely for the purpose of subjecting the quantity to algebraical processes.

From the whole it appears, that the meaning of the signs $+$ and $-$ given in the definitions ought to be steadily adhered to, by which means many of the difficulties of beginners would be avoided.

In dividing a^5 by a^2, we either place the quantities in the form of a fraction, $\frac{a^5}{a^2}$, and expunge like quantities, which gives a^3 for the quotient, or else we subtract the exponent of the divisor from that of the dividend, $a^{5-2} = a^3$. These two methods make the quotients to have in some cases different appearances. Suppose a^2 to be divided by a^5. By the former method $\frac{a^2}{a^5} = \frac{1}{a^3}$. By the second $a^{2-5} = a^{-3}$; so that $a^{-3} = \frac{1}{a^3}$. Here the negative exponent does not represent a negative quantity, but only shows that the quantity placed in the numerator ought to be in the denominator; but in either place it can be subjected with equal ease to all the rules of algebra. From this it appears, that any quantity may be removed from the numerator to the denominator, or from the denominator to the numerator, by changing the sign of its exponent. Thus $\frac{a^2b}{c^2} = a^2bc^{-2}$, and $ab^{-3}c^2 = \frac{ac^2}{b^3}$.

INVOLUTION.

Involution is the method of finding the powers of quantities.

When the quantity is simple, multiply the exponent of each letter by the name of the power to which it is to be raised, and prefix the same power of the coefficient.

If the sign of the quantity be $+$, all its powers are positive;

but if the sign be —, its odd powers have —, and all the rest have +.*

In a fraction, raise its terms separately to the power required.

1. Raise $+3ab^2$ to the 3d power. . Ans. $+27a^3b^6$.

2. . . . $-2a^3x$ to the 6th power. . $+64a^{18}x^6$.

3. . . . $+\frac{4a^3bc^2}{3c}$ to the 5th power. . $+\frac{1024a^{15}b^5c^{10}}{243c^5}$.

4. . . . $-\frac{7a^2}{3b^3}$ to the 3d power. . $-\frac{343a^6}{27b^9}$.

5. . . . $+\frac{2a^{\frac{1}{2}}b^{\frac{3}{4}}c}{3x^{\frac{1}{3}}v^{\frac{1}{2}}}$ to the 8th power. . $+\frac{256a^4b^6c^8}{6561x^{\frac{8}{3}}v^4}$.

When the quantity is compound, raise it by actual multiplication.

6. Thus the powers of $a+b$ are,

2d, $=a^2+2ab+b^2$.

3d, $=a^3+3a^2b+3ab^2+b^3$.

4th, $=a^4+4a^3b+6a^2b^2+4ab^3+b^4$.

5th, $=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

6th, $=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.

The powers of $a-b$ are the same with those of $a+b$, except that the signs of the even terms are —, all the rest are +.

Hence it appears,

1. That the number of terms is one greater than the name of the power.

2. That the exponent of the leading quantity in the first term is the name of the power, and that it decreases by 1 in each of the following terms to the last, where it is 0.

3. That the second quantity is not found in the first term; in the second its exponent is 1; and it increases by 1 in each of the following terms to the last, in which it is the name of the power.

4. That the coefficient of the first term is 1, that of the second is the name of the power, and in the following terms it

* It was shown in Multiplication, that $-x^m \times -x^m = +x^{2m}$, and $+x^{2m} \times -x^m = -x^{3m}$. Hence x^m raised to the nth power $= x^{mn}$, and $-x^m$ raised to the nth power is either $+x^{mn}$ or $-x^{mn}$, according as n is even or odd.

is got by multiplying the coefficient of the preceding term by the exponent of the leading quantity in that term, and dividing the product by the number of that term.

5. That when the signs of both quantities are alike, all the terms have the sign $+$; but if the signs of the quantities be different, the odd terms have $+$, and the even terms $-$.

7. Raise $x-v$ to the 7th power.
Ans. $x^7-7x^6v+21x^5v^2-35x^4v^3+35x^3v^4-21x^2v^5+7xv^6-v^7$.

8. Raise $m-n$ to the 8th power.
Ans. $m^8-8m^7n+28m^6n^2-56m^5n^3+70m^4n^4-56m^3n^5+28m^2n^6-8mn^7+n^8$.

9. Raise $ab-cd$ to the 5th power.
Ans. $a^5b^5-5a^4b^4cd+10a^3b^3c^2d^2-10a^2b^2c^3d^3+5abc^4d^4-c^5d^5$.

10. Raise $2a-3b$ to the 4th power.
Ans. $(2a)^4-4(2a)^3(3b)+6(2a)^2(3b)^2-4(2a)(3b)^3+(3b)^4=16a^4-96a^3b+216a^2b^2-216ab^3+81b^4$.

Note. In this manner care must be taken to distinguish the quantities affected by the different exponents, and to raise them accordingly.

11. Raise $3rs-5vs$ to the 3d power.
Ans. $512r^3s^3-960r^2s^3v+600rs^3v^2-125s^3v^3$.

12. Raise x^2-v^2 to the 5th power.
Ans. $x^{10}-5x^8v^2+10x^6v^4-10x^4v^6+5x^2v^8-v^{10}$.

13. Raise a^2-2ab to the 6th power.
Ans. $a^{12}-12a^{11}b+60a^{10}b^2-160a^9b^3+240a^8b^4-192a^7b^5+64a^6b^6$.

14. Raise $2ac-c^2$ to the 7th power.
Ans. $128a^7c^7-448a^6c^8+672a^5c^9-560a^4c^{10}+280a^3c^{11}-84a^2c^{12}+14ac^{13}-c^{14}$.

15. Raise $3x^2-4xv$ to the 4th power.
Ans. $81x^8-432x^7v+864x^6v^2-768x^5v^3+256x^4v^4$.

16. Raise $5a^2c-3xv^2$ to the 3d power.
Ans. $125a^6c^3-225a^4c^2xv^2+135a^2cx^2v^4-27x^3v^6$.

17. Raise $a+b$ to the nth power.

Ans. $a^n + na^{n-1}b + n.\frac{n-1}{2}a^{n-2}b^2 + n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^3$, &c. or dividing by a^n, and putting A, B, C, &c. for the preceding terms with their signs, it becomes $a^n \times \left(1 + \frac{nb}{a} + \frac{n-1}{2}.\ \frac{bA}{a} + \frac{n-2}{3}.\ \frac{bB}{a} + \frac{n-3}{4}.\ \frac{bC}{a}, \text{ \&c.}\right)$ where the law of continuation is evident.

If the quantity consists of more than two terms, divide the terms into two classes, and raise them as if each class were a simple quantity; after which the classes must be raised according to the exponents placed over them, and then connected with one another, and with the coefficients by multiplication.

18. Raise $a + b - c$ to the 3d power.

Ans. $(a+b)^3 - 3 \times (a+b)^2c + 3(a+b)c^2 - c^3 = a^3 + 3a^2b + 3ab^2 + b^3 - 3a^2c - 6abc - 3b^2c + 3ac^2 + 3bc^2 - c^3$.

19. Raise $a^2 + b^2 - c^2$ to the 2d power.

Ans. $a^4 + 2a^2b^2 + b^4 - 2a^2c^2 - 2b^2c^2 + c^4$.

20. Raise $a^2 - 2ab + b^2$ to the 4th power.

Ans. $a^8 - 8a^7b + 28a^6b^2 - 56a^5b^3 + 70a^4b^4 - 56a^3b^5 + 28a^2b^6 - 8ab^7 + b^8$.

21. Raise $a - b + c - d = (a-b) + (c-d)$ to the 3d power.

Ans. $a^3 - 3a^2b + 3ab^2 - b^3 + 3a^2c - 3a^2d - 6abc + 6aba + 3b^2c - 3b^2d + 3ac^2 - 3bc^2 - 6acd + 6bcd + 3ad^2 - 3bd^2 + c^3 - 3c^2d + 3cd^2 - d^3$.

EVOLUTION.

Evolution is the method of finding the roots of quantities, or those from which given powers have been raised.

In simple quantities, divide the exponents of the letters by the name of the root required, and prefix the same root of the coefficients.

If the sign of the given quantity be $+$, the sign of the root is also $+$. If the sign of the quantity be $-$, the sign of its odd roots is $-$; but it can have no even root, for the square of $+a$, and also of $-a$, is $+a^2$.*

* It was shown in the note on Involution, that x^{mn} is the nth power of x^m, therefore, $x^{\frac{mn}{n}}$ is the nth root of x^{mn}, and consequently that $\frac{1}{n}$ is the proper exponent of the nth root; also that the nth power of $-x^m$ is either $+x^{mn}$ or $-x^{mn}$, according as n is even or odd. Therefore, in the first case, $+x^{\frac{mn}{n}}$, when n is even, may be either $+x^m$ or $-x^m$, and that in this case $-x^{\frac{mn}{n}}$ is impossible.

1. Required the 3d root of a^6b^5. . Ans. a^2b.

2. . . . . 4th root of $\frac{16a^4b^6c^8}{81a^8}$. . $\frac{2ab^{\frac{3}{2}}c^2}{3a^2}$.

3. . . . . 5th root of $\frac{32a^{10}b^8c^5}{c^6x^3}$. . $\frac{2a^2b^{\frac{8}{5}}c}{c^{\frac{6}{5}}x^{\frac{3}{5}}}$.

4. . . . . 6th root of $\frac{m^3n^5}{c^6e^7}$. . . $\frac{m^{\frac{1}{2}}n^{\frac{5}{6}}}{ce^{\frac{7}{6}}}$.

TO FIND THE SQUARE ROOT OF A COMPOUND QUANTITY.

Arrange the terms according to the dimensions of some letter in them, and take the square root of the first term for the first term of the root; subtract its square from the given quantity, and bring down the two next terms to the remainder for a resolvend. Double the root for a divisor, by which divide the first term of the resolvend to get another term of the root; annex this term with its proper sign to the divisor, then multiply the divisor thus completed by it, and subtract the product from the resolvend, and proceed in the same way with the remainder, as in common arithmetic.

1. Required the square root of $x^2-2xv+v^2$.

$$\begin{array}{r|l} & x^2-2xv+v^2\ (x-v \text{ root.} \\ & \underline{x^2\qquad\qquad\quad} \\ 2x-v\,) & \quad -2xv+v^2 \\ & \quad \underline{-2xv+v^2}. \end{array}$$

2. $\sqrt{(x^4-2x^2+1)}$ . . $=x^2-1$.

3. $\sqrt{\left(\frac{x^2}{4}-xv+v^2\right)}$ . . $=\frac{x}{2}-v$.

4. $\sqrt{(x^4-4x^3a+6x^2a^2-4xa^3+a^4)}=x^2-2xa+a^2$.

5. $\sqrt{\left(\frac{a^2}{c^2}-\frac{2ax}{c}+x^2\right)}$ . . $=\frac{a}{c}-x$.

6. $\sqrt{(a^2+2ab+b^2+2ac+2bc+c^2)}=a+b+c$.

7. $\sqrt{(4x^4+6x^3+\frac{89x^2}{4}+15x+25)}=2x^2+\frac{3x}{2}+5$.

8. $\sqrt{(x^6+4x^5+2x^4+9x^2-4x+4)}=x^3+2x^2-x+2$.

TO EXTRACT ANY OTHER ROOT.

Arrange the terms as in Division; take the root of the first term for the first term of the root; raise this root to a power less by one than the given power, and multiply it by the name of the root for a divisor, by which divide the second term of

the given quantity to get another term of the root. Raise the whole root thus found to the given power, and subtract it from the given quantity; if there be a remainder, divide its first term, by the divisor got before, to obtain another term of the root, and proceed as before.

1. Required the cube root of $x^3+3x^2v+3xv^2+v^3$.

$$\begin{array}{rl} & x^3+3x^2v+3xv^2+v^3\big(x+v \text{ root.} \\ & \underline{x^3} \\ 3x^2) & \underline{\quad +3x^2v \qquad\qquad\qquad} \\ (x+v)^3 = & x^3+3x^2v+3xv^2+v^3. \end{array}$$

2. $(27a^3-54a^2c+36ac^2-8c^3)^{\frac{1}{3}} \qquad = 3a-2c.$

3. $(m^6+6m^5-40m^3+96m-64)^{\frac{1}{3}} \qquad = m^2+2m-4.$

4. $(16x^4-96x^3y+216x^2y^2-216xy^3+81y^4)^{\frac{1}{4}}=2x-3y.$

5. $(81a^4-432a^3c+864a^2c^2-768ac^3+256c^4)^{\frac{1}{4}}=3a-4c$

6. $\left(x^5-\frac{5x^4v}{2}+\frac{10x^3v^2}{4}-\frac{10x^2v^3}{8}+\frac{5xv^4}{16}-\frac{v^5}{32}\right)^{\frac{1}{5}} \quad =x-\frac{v}{2}.$

7. $\left(x^6-9x^5+\frac{135x^4}{4}-\frac{135x^3}{2}+\frac{1215x^2}{16}-\frac{729x}{16}+\frac{729}{64}\right)^{\frac{1}{6}}=$ $x-1\frac{1}{2}.$

OF IRRATIONAL QUANTITIES OR SURDS.

IRRATIONAL Quantities or Surds are expressions of the roots of such quantities as are not complete powers.

Thus $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$ is a surd, because a^2 is not a cube.

TO REDUCE SURDS TO A COMMON EXPONENT.

Express them with fractional exponents, and reduce these exponents to a common denominator. This denominator is the common exponent of the root, and the numerators are the exponents of the powers to which the quantities are to be raised.

NOTE. An integer may be expressed as a surd by raising it to any power, and then making the name of the power the exponent of the root: thus $a=a^{\frac{2}{2}}=a^{\frac{4}{4}}$, also $2=\sqrt[3]{8}=\sqrt[5]{32}$.

1. Reduce $\sqrt{3}$ and $\sqrt[3]{2}$ to the same exponent; here $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ $=3^{\frac{3}{6}}$ and $2^{\frac{2}{6}}=\sqrt[6]{27}$ and $\sqrt[6]{4}$.

2. Reduce a and $x^{\frac{1}{4}}$. . Ans. $a^{\frac{4}{4}}$ and $x^{\frac{1}{4}}$.

3. . . . . $\sqrt[4]{15}$ and $\sqrt[6]{9}$. . $\sqrt[12]{3375}$ and $\sqrt[12]{81}$.

4. . . . . $(a+b)^{\frac{1}{2}}$ and $(a-b)^{\frac{1}{3}}$. $(a+b)^{\frac{3}{6}}$ and $(a-b)^{\frac{2}{6}}$.

5. . . . . $(4a)^{\frac{1}{3}}$ and $(3b)^{\frac{1}{4}}$. . $\sqrt[12]{256a^4}$ and $\sqrt[12]{27b^3}$.

6. . . . . $x^{\frac{1}{n}}$ and $v^{\frac{1}{m}}$. . . $x^{\frac{m}{mn}}$ and $v^{\frac{n}{mn}}$.

TO REDUCE A SURD TO ITS MOST SIMPLE FORM.

If any power of the same name with the surd, measures the quantity under the radical sign, place the quotient under the radical, and the root of that power before it for the rational part.

If no such power can be found, the surd is already in its most simple form.

1. Reduce $\sqrt{75}$ to its most simple form. Ans. $5\sqrt{3}$.

2. . . . . $\sqrt[3]{81}$. . . $3\sqrt[3]{3}$.

3. . . . . $\sqrt[3]{243}$ and $\sqrt[3]{16}$. . . $3\sqrt[3]{9}$ and $2\sqrt[3]{2}$.

4. . . . . $\sqrt{98a^2x}$. . . . $7a\sqrt{2x}$.

5. . . . . $(x^3-ax^2)^{\frac{1}{2}}$. . . . $x(x-a)^{\frac{1}{2}}$.

6. . . . . $(a^4x+3a^3x^2)^{\frac{1}{3}}$. . . $a\sqrt[3]{x}\times(a+3x)^{\frac{1}{3}}$.

7. . . . . $(32a^6-96a^5x)^{\frac{1}{5}}$. . . $2a(a-3x)^{\frac{1}{5}}$.

TO ADD AND SUBTRACT SURDS.

Reduce them to the same exponent, and to their most simple forms: then, if the quantity under the radical sign be the same in them all, add or subtract the rational parts, and to the sum or difference annex the common surd. But if the quantities under the radical be different, the surds must be added or subtracted as unlike quantities.

1. Add $3\sqrt{2}$ and $2\sqrt{2}$. . Ans. $5\sqrt{2}$.

2. . . . $ab^{\frac{1}{3}}$ and $\frac{3ab^{\frac{1}{3}}}{2}$. . . $\frac{5ab^{\frac{1}{3}}}{2}$.

3. . . . $\sqrt[3]{48a^7}$ and $\sqrt[3]{6a}$. . . $(2a^2+1)\sqrt[3]{6a}$.

4. . . . $2\sqrt{a^2x}$ and $3\sqrt{64x^3}$. . $(2a+24x)\sqrt{x}$.

5. From $9a\sqrt{3}$ take $2a\sqrt{3}$. . $7a\sqrt{3}$.

6. . . . $\sqrt[3]{81a}$ take $\sqrt[3]{24a}$. . $\sqrt[3]{3a}$.

7. From $2\sqrt{50}$ take $\sqrt{18}$. . Ans. $7\sqrt{2}$.

8. . . . $\sqrt{80a^4x}$ take $\sqrt{20a^2x^3}$. . $(4a^2-2ax)\sqrt{5x}$.

9. Add and subtract $3\sqrt{\frac{5}{27}}$, $4\sqrt{\frac{3}{5}}$. . $\frac{17}{15}\sqrt{15}$ and $\frac{7}{15}\sqrt{15}$.

TO MULTIPLY AND DIVIDE SURDS.

Reduce them to a common exponent, if they have different ones, and then find the product or quotient of the rational parts, and also of the surds; and the two joined together, with the common radical sign between them, will give the whole product or quotient required.

NOTE. When the quantities under the radical signs are alike, the product or quotient of the surds is found by adding or subtracting their exponents.

1. Multiply $\sqrt{2}$ by $\sqrt[3]{2}$. . Ans. $\sqrt[6]{32}$.
2. . . . . $\sqrt[3]{4}$ by $\sqrt[3]{5}$. . . $\sqrt[3]{20}$.
3. . . . . $a^{\frac{1}{3}}$ by $a^{\frac{3}{4}}$. . . $a\sqrt[12]{a}$.
4. . . . . $a^{\frac{1}{2}}$ by $b^{\frac{2}{3}}$. . . $\sqrt[6]{a^3b^4}$.
5. . . . . $2\sqrt{3}$ by $3\sqrt[3]{4}$. . . $6\sqrt[6]{432}$.
6. Divide $\sqrt{7}$ by $\sqrt[4]{7}$. . . $\sqrt[4]{7}$.
7. . . . . $\sqrt[3]{8}$ by $\sqrt[3]{2}$. . . $\sqrt[3]{4}$.
8. . . . . $a^{\frac{3}{4}}b^{\frac{1}{2}}$ by $a^{\frac{2}{3}}b^{\frac{1}{3}}$. . . $a^{\frac{1}{12}}b^{\frac{1}{6}}$.
9. . . . . $2\sqrt[3]{bc}$ by $3\sqrt{ac}$. . . $\frac{2}{3}\sqrt[6]{\frac{b^2}{a^3c}}$.
10. . . . . $10\sqrt[3]{108}$ by $5\sqrt[3]{84}$. . $\frac{2}{7}\sqrt[3]{441}$.

INVOLUTION AND EVOLUTION OF SURDS.

The powers and roots of surds are found as those of other quantities, by multiplying or dividing their exponents by the name of the power or root.

In some cases it is preferable to raise the quantity under the radical to the power or root required, and then to place the radical sign over it.

1. The 4th power of $\sqrt{3a}$ $= 9a^2$.
2. The 3d power of $(a-b)^{\frac{1}{3}} = a-b$.
3. The 4th power of $\frac{1}{6}\sqrt{6}$ $= \frac{1}{36}$.

4. The 5th power of $\frac{2\sqrt{a}}{3\sqrt[5]{c}}$ $= \frac{32\sqrt{a^5}}{243c}$.

5. The 3d root of $a^{\frac{1}{2}}b^{\frac{3}{4}}c$ $= a^{\frac{1}{6}}b^{\frac{1}{4}}c^{\frac{1}{3}}$.

6. The 4th root of $\frac{ab^{\frac{1}{2}}c^4}{a^3}$ $= \frac{a^{\frac{1}{4}}b^{\frac{1}{8}}c}{a^{\frac{3}{4}}}$.

7. The 3d root of $\frac{1}{8}\sqrt{2}$ $= \frac{1}{2}\sqrt[6]{2}$.

8. The 5th root of $\frac{b^{\frac{2}{3}}}{32a^{\frac{1}{3}}}$ $= \frac{1}{2}\sqrt[15]{\frac{b^2}{a}}$.

TO FIND THE SQUARE ROOT OF A COMPOUND SURD.

When a quantity consists of two terms, a rational and a surd; if it has a root, the rational part is the sum of the squares of its terms, and the surd is the double of their product.

From the square of the rational term subtract the quantity affected by the radical sign, and take the square root of the remainder; add it to the rational term, and also subtract it from that term, and take the halves of the sum and remainder for the squares of the two terms of the root.

1. $(6-\sqrt{20})^{\frac{1}{2}} = \sqrt{5}-1$, for $\sqrt{36-20} = \sqrt{16} = 4$, and $\sqrt{\frac{6\pm 4}{2}} = \sqrt{5}$ and 1.

2. $(136-96\sqrt{2})^{\frac{1}{2}} = 6\sqrt{2}-8$.

3. $(51-10\sqrt{2})^{\frac{1}{2}} = 5\sqrt{2}-1$.

4. $(14-6\sqrt{5})^{\frac{1}{2}} = 3-\sqrt{5}$.

5. $(5-2\sqrt{6})^{\frac{1}{2}} = \sqrt{3}-\sqrt{2}$.

6. $(76-42\sqrt{3})^{\frac{1}{2}} = 7-3\sqrt{3}$.

7. $(19+8\sqrt{3})^{\frac{1}{2}} = 4+\sqrt{3}$.

8. $(12-2\sqrt{35})^{\frac{1}{2}} = \sqrt{7}-\sqrt{5}$.

9. $(7+4\sqrt{3})^{\frac{1}{2}} = 2+\sqrt{3}$.

10. $(7-2\sqrt{10})^{\frac{1}{2}} = \sqrt{5}-\sqrt{2}$.

11. $(39-6\sqrt{30})^{\frac{1}{2}} = \sqrt{30}-3$.

EQUATIONS.

WHEN two expressions are equal to one another, they are written with the sign $=$ of equality between them, and the whole is called an equation. Thus $x-a=b+c$ is an equation; $x-a$ is called the left side, and $b+c$ the right side of the equation.

An equation which contains only the first power of the unknown quantity or quantities is called a simple equation.

RESOLUTION OF SIMPLE EQUATIONS CONTAINING ONLY ONE UNKNOWN QUANTITY.

The resolution of simple equations containing one unknown quantity consists in separating the unknown quantity from the other quantities with which it is connected, and making it stand alone upon one side of the equation, and the known quantities upon the other side. This is performed by the following rules taken in their order:

RULE 1. If a term be divided by any quantity, multiply every term by the divisor.

In this way the equation may be cleared of fractions.

RULE 2. Any term may be transposed from one side of the equation to the other, by changing its sign from $+$ to $-$, or from $-$ to $+$.

In this way the terms containing the unknown quantity may be brought to one side of the equation, and the known terms to the other; after which they may be collected by addition.

COR. If a term be found on both sides with the same sign, it may be erased from both.

RULE 3. If the unknown quantity be multiplied by any other, divide both sides by the multiplier.

In this way the value of the unknown quantity is found, when there are no surds nor powers.

RULE 4. If the equation have a surd in it, after bringing it to one side by itself, take away the radical sign, and raise the other side to the corresponding power.

RULE 5. If one side of the equation be a complete power, take the corresponding root of both sides.*

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* It is evident that the operations prescribed in these rules do not render the two sides of the equation unequal, for they are both increased or diminished in the same degree. Thus, in the first operation, both sides are multiplied by the same quantity; in transposition the same quantity is subtracted from both sides; in the third both sides are divided by the same quantity; in the fourth they are both raised to the same power; and in the last the same root is taken of both sides.

Let the equation be $2x - \frac{19}{4} = \frac{3x}{4} + 4$

Multiply by 4, . . $8x - 19 = 3x + 16$
Add $19 - 3x$ to both sides, $8x - 3x = 16 + 19$
And collecting, . . . $5x = 35$
Divide by 5, . . . $x = 7$
So that 7 is the value of x.

In the second line the equation is cleared of fractions, and in the third line the quantities 19 and $3x$ are transposed with their signs changed; and it is evident that the two sides of the equation have been kept equal to one another in every line.

Let the equation be $(3x+1)^{\frac{1}{2}} + 5 = 10$

By transposing 5, $(3x+1)^{\frac{1}{2}} = 10 - 5 = 5$
Square by rule 4, $3x + 1 = 25$
Transposing 1, . . $3x = 25 - 1 = 24$
And dividing by 3, . $x = 8$.

The removal of the sign from the radical is equivalent to the raising of it to the power.

Let the equation be . $9x^2 + 9 = 3x^2 + 63$
By transposing, . $9x^2 - 3x^2 = 63 - 9$
Collecting, . . . $6x^2 = 54$
Dividing by 6, . . $x^2 = 9$
Taking the square root, . $x = 3$.

Any analogy or proportion may be changed into an equation by making the product of the first and last terms equal to the product of the two mean terms.

Let $2 + x : 6 - x :: 15 : 9$
Then $9(2+x) = 15(6 - x$
Or $18 + 9x = 90 - 15x$
Transposing, . $9x + 16x = 90 - 18$
And collecting, . $24x = 72$
Dividing by 24, . $x = 3$.

Again, let $x - 5 : 2x :: 5 : 20$
Then $20 \times (x - 5) = 5 \times 2x$
Or $20x - 100 = 10x$
Transposing, $20x - 10x = 100$
And collecting, . $10x = 100$
Dividing by 10, . $x = 10$.

RESOLVE THE FOLLOWING EQUATIONS:

EQUATIONS.	ANSWERS.
1. $5x + 3 = 2x + 15$.	$x = 4$.
2. $24 - 2x = 3x - 6$.	$x = 6$.

EQUATIONS.	ANSWERS.
3. $15x-26=12x+16$.	$x=14$.
4. $\frac{x}{2}-3=5$.	$x=16$.
5. $6-x=4-\frac{2x}{3}$.	$x=6$.
6. $4x-8=3x+20$.	$x=28$.
7. $40-6x-16=120-14x$.	$x=12$.
8. $x+\frac{x}{2}+\frac{x}{3}=11$.	$x=6$.
9. $ax+2ab=3c^2$.	$x=\frac{3c^2}{a}-2b$.
10. $5ax-3b=2dx+c$.	$x=\frac{3b+c}{5a-2d}$.
11. $2x-\frac{x}{2}+1=5x-2$.	$x=\frac{6}{7}$.
12. $x^{\frac{1}{2}}-2=6$.	$x=64$.
13. $(4x+16)^{\frac{1}{2}}=12$.	$x=32$.
14. $5x-15=2x+6$.	$x=7$.
15. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=10$.	$x=9\frac{3}{13}$.
16. $3x^2-x=8x+x^2$.	$x=4\frac{1}{2}$.
17. $x-a=\frac{x^2}{x-a}$.	$x=\frac{a}{2}$.
18. $(2x+3)^{\frac{1}{3}}+4=8$.	$x=30\frac{1}{2}$.
19. $\left(\frac{2x}{3}\right)^{\frac{1}{2}}+5=7$.	$x=6$.
20. $\frac{x-3}{2}+\frac{x}{5}=20-\frac{x-19}{2}$.	$x=25\frac{5}{6}$.
21. $\frac{a}{1+x}+\frac{a}{1-x}=b$.	$x=\left(\frac{b-2a}{b}\right)^{\frac{1}{2}}$.
22. $x+(a^2+x^2)^{\frac{1}{2}}=\frac{2a^2}{(a^2+x^2)^{\frac{1}{2}}}$.	$x=\frac{a}{\sqrt{3}}$.
23. $x^{\frac{1}{2}}+(a+x)^{\frac{1}{2}}=\frac{2a}{(a+x)^{\frac{1}{2}}}$.	$x=\frac{a}{3}$.
24. $(12+x)^{\frac{1}{2}}=2+x^{\frac{1}{2}}$.	$x=4$.

EQUATIONS.	ANSWERS.
25. $(a^2+x^2)^{\frac{1}{2}}=(b^4+x^4)^{\frac{1}{4}}$.	$x=\left(\frac{b^4-a^4}{2a^2}\right)^{\frac{1}{2}}$.
26. $bx^2+c+3=2bx^2+1$.	$x=\left(\frac{c+2}{b}\right)^{\frac{1}{2}}$.
27. $4x-\frac{x-1}{2}=x+\frac{2x-2}{5}+24$.	$x=11$.
28. $a+x=[a^2+x(b^2+x^2)^{\frac{1}{2}}]^{\frac{1}{2}}$.	$x=\frac{b^2}{4a}-a$.
29. $\frac{3x}{a}-c+\frac{x}{b}=4x+\frac{2x}{d}$.	$x=\frac{abcd}{(3b+a)d-2ab(2d+1)}$.
30. $3x-\frac{a}{b}-cx=\frac{a+x}{3}-\frac{b-x}{a}$.	$x=\frac{a^2b-3b^2+3a^2}{8ab-3abc-3b}$.
31. $5ax-2b+4bx=2x+5c$.	$x=\frac{5c+2b}{5a+4b-2}$.

RESOLUTION OF SIMPLE EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

When there are several unknown quantities, there must be as many independent equations involving them: and from these an equation must be deduced, which contains only one of the unknown quantities.

This may be performed by any of the following rules:

Rule I. Find a value of one of the unknown quantities in each of the equations, supposing all the rest to be known. Make these values equal to one another, and from them find the value of another unknown quantity. Make again these values equal, and find another unknown quantity, and so on, until an equation be obtained containing only one unknown quantity, which is to be resolved by the preceding rules.

Rule II. Find a value of one of the unknown quantities in that equation in which it is least involved; substitute this value and its powers for that unknown quantity and its powers in all the other equations, and proceed in the same way with these equations to get rid of other unknown quantities.

Rule III. Multiply the equations by such quantities as will make the coefficients of one of the unknown quantities, or of its highest power, the same in all the equations; then, if the signs of these equal terms be like, subtract the equations, but if the signs be unlike, add them, and new equations will arise, wanting that unknown quantity or its highest power, and these equations are to be treated in the same way.

NOTE. The first method seems to be the most regular; the second is shorter than the first, but the reductions are more intricate; the third is the most simple and expeditious.

Let the equations be $\left.\begin{array}{r} x+\ y=12, \\ \text{and } 5x+3y=50 \end{array}\right\}$ To find the values of x and y.

By RULE I.

From the 1st equation $x=12-y$, and from the 2d $x=\frac{50-3y}{5}$

$\therefore \quad 12-y=\frac{50-3y}{5}$, clearing this equation of fractions $60-5y=50-3y$, transposing and collecting $10=2y$ or $y=5$, and $x=12-y=7$.

By RULE II.

From the 1st equation $x=12-y$; substituting this value for x in the 2d equation, we have

$$5(12-y)+3y=50$$

Or $\quad 60-5y+3y=50$

$\therefore \quad 10=2y$, or $y=5$, and $x=12-y=7$.

By RULE III.

1st Equation multiplied by 5,	$5x+5y=60$
2d Equation, ...	$5x+3y=50$
By subtraction, . . .	$2y=10$
	$\therefore\ y=5$.

Let the equations be $\left.\begin{array}{r} x+y\ +z\ =\ 53 \\ x+2y+3z=105 \\ x+3y+4z=134 \end{array}\right\}$ To find the values of x, y, and z.

By RULE I.

From the 1st equation $x=53-y-z$, from the 2d $x=105-2y-3z$, and from the 3d $x=134-3y-4z$; whence

$$53-y-z=105-2y-3z$$
$$53-y-z=134-3y-4z,$$

and these equations, by transposing and collecting, become

$$y+2z=52$$
$$2y+3z=81.$$

Now from the 1st of these $y=52-2z$, and from the 2d $y=\frac{81-3z}{2}$; whence $52-2z=\frac{81-3z}{2}$, an equation containing only one unknown quantity, which, by clearing of fractions, transposing and collecting, gives $z=23$; hence $y=52-2z=52-46=6$, and $x=53-y-z=53-29=24$.

By Rule II.

From the 1st equation $x = 53 - y - z$, and this value bstituted for x in the 2d and 3d equations gives

$$53 - y - z + 2y + 3z = 105, \text{ or } \quad y + 2z = 52 \text{ (A)},$$

$$\text{and } 53 - y - z + 3y + 4z = 134, \text{ or } 2y + 3z = 81 \text{ (B)}.$$

ow from equation (A) $y = 52 - 2z$, and this substituted for in equation (B) gives $2(52 - 2z) + 3z = 81$, an equation ntaining only one unknown quantity; whence, by transpos-g and collecting, we obtain $z = 23$, and the values of x and as before.

By Rule III.

2d Equation,	. .	$x + 2y + 3z = 105$
1st Equation,	. .	$x + y + z = 53$
By subtraction,	.	$y + 2z = 52$ (A).
3d Equation,	.	$x + 3y + 4z = 134$
2d Equation,	. .	$x + 2y + 3z = 105$
By subtraction,	.	$y + z = 29$ (B).
Equation (A),	. .	$y + 2z = 52$
Equation (B),	. .	$y + z = 29$
By subtraction,	. .	$z = 23$.

	EQUATIONS.	ANSWERS.
2.	$\left.\begin{array}{l} 5x + 8y = 124 \\ 3x - 2y = 20 \end{array}\right\}$.	$\left\{\begin{array}{l} x = 12 \\ y = 8. \end{array}\right.$
3.	$\left.\begin{array}{l} 5x - 3y = 90 \\ 2x + 5y = 160 \end{array}\right\}$.	$\left\{\begin{array}{l} x = 30 \\ y = 20. \end{array}\right.$
4.	$\left.\begin{array}{l} x - y = 2 \\ 8y + 5x - 6y = 120 \end{array}\right\}$.	$\left\{\begin{array}{l} x = 17\frac{5}{7} \\ y = 15\frac{5}{7}. \end{array}\right.$
5.	$\left.\begin{array}{l} \frac{x}{2} + \frac{y}{3} = 16 \\ \frac{x}{5} - \frac{y}{9} = 2 \end{array}\right\}$.	$\left\{\begin{array}{l} x = 20 \\ y = 18. \end{array}\right.$
6.	$\left.\begin{array}{l} x + y = a \\ x^2 - y^2 = b \end{array}\right\}$.	$\left\{\begin{array}{l} x = \frac{1}{2}a + \frac{b}{2a} \\ y = \frac{1}{2}a - \frac{b}{2a}. \end{array}\right.$
7.	$\left.\begin{array}{l} 4x + 3y = 31 \\ 3x + 2y = 22 \end{array}\right\}$.	$\left\{\begin{array}{l} x = 4 \\ y = 5. \end{array}\right.$
8.	$\left.\begin{array}{l} 5x - 4y = 19 \\ 4x + 2y = 36 \end{array}\right\}$.	$\left\{\begin{array}{l} x = 7 \\ y = 4. \end{array}\right.$

EQUATIONS.	ANSWERS.
9. $\left.\begin{array}{l} 3x+7y=79 \\ 2y-\frac{x}{2}=9 \end{array}\right\}$.	$\left\{\begin{array}{l} x=10 \\ y=7. \end{array}\right.$
10. $\left.\begin{array}{l} \frac{x+y}{3}+1=6 \\ \frac{x-y}{7}+3=4 \end{array}\right\}$.	$\left\{\begin{array}{l} x=11 \\ y=4. \end{array}\right.$
11. $\left.\begin{array}{l} \frac{x+y}{3}-2y=2 \\ \frac{2x-4y}{5}+y=\frac{23}{5} \end{array}\right\}$.	$\left\{\begin{array}{l} x=11 \\ y=1. \end{array}\right.$
12. $\left.\begin{array}{l} \frac{3x-7y}{3}=\frac{2x+y+1}{5} \\ 8-\frac{x-y}{5}=6 \end{array}\right\}$.	$\left\{\begin{array}{l} x=13 \\ y=3. \end{array}\right.$
13. $\left.\begin{array}{l} x+y=13 \\ x+z=14 \\ y+z=15 \end{array}\right\}$.	$\left\{\begin{array}{l} x=6 \\ y=7 \\ z=8. \end{array}\right.$
14. $\left.\begin{array}{l} 2x+3y+4z=29 \\ 3x+2y+5z=32 \\ 4x+3y+2z=25 \end{array}\right\}$.	$\left\{\begin{array}{l} x=2 \\ y=3 \\ z=4. \end{array}\right.$
15. $\left.\begin{array}{l} x+100=y+z \\ y+100=2x+2z \\ z+100=3x+3y \end{array}\right\}$.	$\left\{\begin{array}{l} x=9\frac{1}{11} \\ y=45\frac{5}{11} \\ z=63\frac{7}{11}. \end{array}\right.$
16. $\left.\begin{array}{l} x+y=90-z \\ 2x+40=3y+20 \\ x+20=2z+5 \end{array}\right\}$.	$\left\{\begin{array}{l} x=35 \\ y=30 \\ z=25. \end{array}\right.$
17. $\left.\begin{array}{l} x+y=a \\ x+z=b \\ y+z=c \end{array}\right\}$.	$\left\{\begin{array}{l} x=\frac{b+a-c}{2} \\ y=\frac{a+c-b}{2} \\ z=\frac{c+b-a}{2}. \end{array}\right.$
18. $\left.\begin{array}{l} \frac{1}{2}x+\frac{1}{3}v+\frac{1}{4}z=62 \\ \frac{1}{3}x+\frac{1}{4}v+\frac{1}{5}z=47 \\ \frac{1}{4}x+\frac{1}{5}v+\frac{1}{6}z=38 \end{array}\right\}$.	$\left\{\begin{array}{l} x=24 \\ v=60 \\ z=120. \end{array}\right.$

QUADRATIC EQUATIONS.

If, after all the unknown quantities, except one, are exter nated from an equation, both that unknown quantity and square are found in it, the equation is called a Quadratic.

RESOLUTION OF QUADRATIC EQUATIONS.

Having cleared the equation, and brought the terms involving the unknown quantity to one side of it by themselves, divide by the coefficient of the square of the unknown quantity, if it have one; then add to both sides the square of half the coefficient of the unknown quantity, which will complete the square of the side containing the unknown quantity; after which extract the square root of both sides, and the equation will be reduced to a simple one, which may be resolved as before.

NOTE 1. Since the square root of $x^2-2ax+a^2$ is either $a-x$ or $x-a$, the root of the known side of the equation must have both the signs + and — before it. Sometimes both these give proper solutions, and at other times only one of them.

NOTE 2. The root of the side involving the unknown quantity consists of that quantity, and of $\frac{1}{2}$ its coefficient with its sign.*

Let the equation be $3x^2+12x=96$
By dividing by 3, . $x^2+4x=32$
Add the square of 2, $x^2+4x+4=36$
And taking the root, . $x+2=\pm 6$
And transposing, . . $x=\pm 6-2=+4$ or -8.
Here the positive value of the root only is proper.

Let the equation be $2x^2-8x=90$
Dividing by 2, . $x^2-4x=45$
Completing the square, $x^2-4x+4=49$
Taking the root, . $x-2=\pm 7$
Transposing, . . . $x=\pm 7+2=+9$ or -5.
Here also the root 7 is greater than $\frac{1}{2}$ the coefficient of x; therefore the positive value only is proper.

Let the equation be $15x-x^2=54$
Or $x^2-15x=-54$
Completing the square, $x^2-15x+\frac{225}{4}=\frac{225}{4}-54=+\frac{9}{4}$
Taking the root, . . $x-\frac{15}{2}=\pm\frac{3}{2}$
Transposing, . . . $x=+\frac{15}{2}\pm\frac{3}{2}=+9$ or $+6$.

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* Quadratic equations assume one of these three forms, viz. $x^2+ax=+b$; $x^2-ax=+b$; or $x^2-ax=-b$; and when they are resolved by the rule, the value of x assumes one of these forms, $x=\frac{-a\pm\sqrt{a^2+4b}}{2}$; $x=\frac{+a\pm\sqrt{a^2+4b}}{2}$; or $x=\frac{+a\pm\sqrt{a^2-4b}}{2}$.
If a positive answer is required, the sign of the radical in the first two forms

Here both the roots are proper. But it is to be remarked, that if 54 had been greater than $\frac{225}{4}$, the known side would have been negative, and its root impossible; in which case x would have had no value in numbers.

NOTE. To avoid fractions, instead of dividing by the coefficient of x^2, and then adding the square of $\frac{1}{2}$ the coefficient, multiply the equation by 4 times the coefficient of x^2, and then add the square of the coefficient, which x had before multiplying.

Let the equation be $7x^2-20x=32$
Multiplying by $4\times7=28$, $196x^2-560x=896$
Adding $400=20^2$, . $196x^2-560x+400=1296$
Taking the root, . . . $14x-20=\pm36$
Whence . . . . . . $x=+4$ or $-1\frac{1}{7}$.

EQUATIONS.	ANSWERS.
1. $x^2+6x=27$.	$x=+3$.
2. $x^2+10x=56$.	$x=4$.
3. $x^2-4x=60$.	$x=10$.
4. $x^2-6x=72$.	$x=12$.
5. $8+x^2-6x=80$.	$x=12$.
6. $8x-20=70-2x^2$.	$x=5$.
7. $3x^2+6=3x+5\frac{1}{3}$.	$x=\frac{2}{3}$ or $\frac{1}{3}$.
8. $\frac{x}{3}+42\frac{2}{3}=\frac{x^2}{2}+20\frac{1}{2}$.	$x=7$.
9. $3x^2-9=76-2x$.	$x=5$.
10. $x^2-x=210$.	$x=15$.
11. $\frac{1}{2}x^2+7\frac{3}{8}=\frac{1}{3}x+8$.	$x=1\frac{1}{2}$.
12. $4x^2-3x=85$.	$x=5$.
13. $\frac{4x^2}{3}-11=\frac{x}{3}$.	$x=3$.
14. $5x^2+4x=273$.	$x=7$.
15. $\frac{7}{x+1}+\frac{2}{x}=5$.	$x=\frac{2+\sqrt{14}}{5}$.

must be +, but in the third it may be either + or —. There is, however, a limitation in this case, for $4b$ must not be greater than a^2, otherwise the quantity below the radical sign would be negative, and its root impossible. This happens when the absolute term b is greater than $\frac{1}{4}a^2$, the square of $\frac{1}{2}$ the coefficient of x.

EQUATIONS.	ANSWERS.
16. $\frac{x}{5}+\frac{5}{x}=5\frac{1}{5}$. . .	$x=25$ or 1.
17. $\frac{3x}{x+2}-\frac{x-1}{6}=x-9$.	$x=10$.
18. $x^2+6ax=c^2$. .	$x=(c^2+9a^2)^{\frac{1}{2}}-3a$.
19. $\frac{x}{a}+\frac{a}{x}=\frac{2}{a}$. . . .	$x=1\pm\sqrt{1-a^2}$.

NOTE. If the equation contain two powers of the unknown quantity, and the exponent of the one is double that of the other, it may be resolved like a quadratic.

20. Let the equation be $x^6-6x^3=16$
Completing the square, $x^6-6x^3+9=25$
Taking the root, . . $x^3-3=\pm 5$
Transposing, . . . $x^3=3\pm 5=8$
Taking the cube root, . . $x=2$.

EQUATIONS.	ANSWERS.
21. $2x^4-x^2=496$. .	$x=4$.
22. $x^4+2ax^2=b$. . .	$x=(\sqrt{a^2+b}-a)^{\frac{1}{2}}$.
23. $x-8x^{\frac{1}{2}}=9$. . .	$x=81$ or 1.
24. $x-x^{\frac{1}{2}}=a$. . .	$x=a+\frac{1}{2}\pm\sqrt{a+\frac{1}{4}}$.
25. $\frac{1}{2}x-\frac{1}{3}x^{\frac{1}{2}}=22\frac{1}{6}$. .	$x=49$ or $40\frac{1}{9}$.
26. $(1+x)^{\frac{1}{2}}-2(1+x)^{\frac{1}{4}}=4$.	$x=55\pm 24\sqrt{5}$.
27. $3x^{2n}-2x^n=25$. .	$x=\left(\frac{1\pm 2\sqrt{19}}{3}\right)^{\frac{1}{n}}$.
28. $x^n-6x^{\frac{n}{2}}=e$. . .	$x=(18+e\pm 6\sqrt{e+9})^{\frac{1}{n}}$.
29. $4ax^4-bx^2=c$. . .	$x=\left(\frac{b\pm\sqrt{16c+b^2}}{8a}\right)^{\frac{1}{2}}$.

SOLUTION OF QUESTIONS.

WHEN a question is proposed, the analyst ought to form a clear idea of its nature, and then attempt to express its terms, and the relations of its parts, in algebraical characters, putting the letters x, y, z, &c. for the unknown quantities in it; and

in this way he must deduce as many independent equations from the conditions of the question as there are unknown quantities in it, which he can always do when the question is properly limited; after which, these equations being resolved by the preceding rules, will give the answer or answers.

Put x for the greatest unknown quantity, y for the next, z, v, &c. for the lesser ones in their order.

Suppose it to be a condition of the question, that

The two quantities together, or their sum, amounts to 18. This condition may be expressed thus, . $x+y=18$

Their excess, difference, &c. is 6, . $x-y=6$

Their product, rectangle, the one into the other, or multiplied by it, is 72, . . $xy=72$

One of them taken out of the other, divided by it, applied to it, or their quotient, is 2, $\frac{x}{y}=2$

The greater is to the less, or their ratio is as 4 to 2, ... $x:y::4:2$

And this proportion, by multiplying the means together, and also the extremes, becomes an equation, . . . $2x=4y$

The sum of their squares is 180, . $x^2+y^2=180$

The difference of their squares is 108, $x^2-y^2=108$

And in a similar way may any other relations of the quantities be expressed in equations.

When the relation of one unknown quantity to another is simple, a letter may be taken for one of them, and an expression for the other deduced from the relation between them, which will abridge the work, and render it more elegant. Thus, if their difference be 3, take y for the less, and $y+3$ will be the greater.

It will often abridge the work, if letters are taken not for the unknown quantities themselves, but for their sum, difference, or any other relation from which the quantities may be easily found.

QUESTIONS PRODUCING SIMPLE EQUATIONS.

1. To find such a number, that, if it be multiplied by 5, and also by 3, the former product shall exceed the latter by 26. Let $x=$ the number required, then the first product is $5x$, the second $3x$, and their difference $5x-3x=26$, or $2x=26 \therefore x=13$.

2. To find a number, to which if 27 be added, the sum shall be 10 times the number required. Let $x=$ the number required, then $10x=x+27$, or $9x=27 \therefore x=3$.

3. To find a number, from which if 4 be taken, and the remainder multiplied by 3, the product shall be twice the

number sought. Let $x =$ the number required, then $(x-4)3 = 2x$, or $3x - 12 = 2x$; whence $3x - 2x = 12$, or $x = 12$.

4. To find a number of which the fourth part exceeds the fifth part by 13.

$\frac{x}{4} - \frac{x}{5} = 13.$ Ans. 260.

5. To find a number, to the half of which if 7 be added, the sum shall be equal to twice the number with 20 taken from it. Ans. 18.

6. To find a number, of which the square shall be equal to 4 times the number, together with 5 times the same number. Ans. 9.

7. To find a number, to which if its half, its third, and its fourth parts be added, the sum shall be equal to the square of that number.

$x^2 = x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4}.$ Ans. $2\frac{1}{12}$.

8. To find a number, from which if 3 be taken, and the remainder multiplied by 3, and then 4 added to the product, the sum divided by 5 shall give half the number sought. Ans. 10.

9. To find a number of pounds, to which if 3 be added, and the sum multiplied by 12, the product shall be equal to the number of shillings in the value of the pounds, diminished by as many crowns as there are pounds required.

$(x+3)12 = 20x - 5x.$ Ans. £12.

10. To find two numbers, of which the sum is 133, and their difference 47. Ans. 90 and 43.

11. To find two numbers, of which the sum is 84, and their quotient 13. Ans. 78 and 6.

12. To find two numbers, of which the difference is 104, and their quotient 9. Ans. 117 and 13.

13. To find two numbers, so that 3 times the greater added to twice the less shall make 54, and 4 times the greater with 3 times the less shall make 75. Ans. 12 and 9.

14. To find two numbers, so that the greater with half the less shall make 25, and the less with half the greater shall make 23. Ans. 18 and 14.

15. To find two numbers in the ratio of 4 to 3, so that if one be added to each of them, the sums shall be in the ratio of 9 to 7.

$3x = 4y$, $(x+1) \times 7 = (y+1) \times 9.$ Ans. 8 and 6.

16. To find two numbers, of which the difference shall be 9, and the difference of their squares 351. Ans. 24 and 15.

17. To divide the number 36 into two parts, so that the square of the greater part shall exceed that of the less by 360. Ans. 23 and 13.

18. To divide the number 72 into two parts, so that three times the greater shall exceed twice the less by 121. Ans. 53 and 19.

19. To divide the number 56 into two parts, which shall be to one another as 4 to 3. Ans. 32 and 24.

20. To find a number, so that its half added to its third part shall be greater by $6\frac{1}{2}$ than its double divided by 5. Ans. 15.

21. To find a number, from the double of which if 22 be taken, the remainder shall exceed 100 as much as the number itself is below 100.

$$2x - 22 - 100 = 100 - x. \qquad \text{Ans. } 74.$$

22. A person being asked his age, replied, that $\frac{1}{5}$ of his age, multiplied by $\frac{1}{6}$ of his age, would produce his age. How old was he? Ans. 30.

23. A general sends out $\frac{1}{3}$ of his army, and 1500 men more, and he retains $\frac{1}{2}$ of his army, and 1200 men more. How many men had he in his army? Ans. 16200.

24. A gentleman distributing money among some poor people, found that he wanted 10s. to be able to give 5s. to each of them; he therefore gave each 4s., and then he had 5s. left. How much money had he, and how many poor were there? Ans. 65s., 15 poor.

25. To find two numbers in the ratio of 3 to 2, so that their sum shall be the sixth part of their product. Ans. 15 and 10.

26. There were 6 children in a family, whose ages differed by 2 years, and each received a guinea for every year of his age, the money they received amounted to 72 guineas. Required their ages? Ans. 7 youngest, 17 eldest.

27. A and B inherited equal estates; but A spent annually £60 more than his income, while B saved £80 annually; in consequence of which, at the end of 12 years, B was twice as rich as A. Required the value of their estates?

$$(x - 60 \times 12)2 = x + 80 \times 12. \qquad \text{Ans. £}2400.$$

28. A says to B, If you will give me £25, I shall have as much money as you shall have left. Says B, If you give me £30, I shall then have twice as much as you will have remaining. How much had each? Ans. B £190, A £140.

29. A farmer has 15 more cows than horses, and as many

scores of sheep as horses and cows together; the number of all the three is 651. How many has he of each kind?

Ans. 8 horses, 23 cows, 31 scores sheep.

30. Two merchants join in company with a capital of £2000. A's share was 11 months in trade, and B's 9 months, and their shares of the gain were equal. What was the stock of each? Ans. B's £1100, A's £900.

31. A field was sown with wheat at 35s. per boll, and produced 9 returns: the crop was sold at 30s. per boll, and, after paying for the seed, there remained £293, 15s. How much wheat was sown? Ans. 25 bolls.

32. A merchant laid aside £200 annually for his expenses, and increased his capital annually by $\frac{1}{3}$ of what was not thus expended. At the end of three years his capital was double of what he began with. What was it at first?

$$x+\frac{x-800}{3}+\frac{4x-3200}{9}+\frac{16x-12800}{27}=2x. \qquad \text{Ans. £2960.}$$

33. Five persons have money divided among them. The share of the first was £10 more than that of the second; the share of the second was £16 less than that of the third; the share of the third was £5 more than that of the fourth; and the share of the fourth £15 less than that of the fifth: also the shares of the two last were together equal to the sum of the shares of the other three. What was the share of each?

Ans. £21, £11, £27, £22, £37.

34. Two travellers set out at the same time to meet one another, from two places distant 390 miles: the first travels 30 miles in a day, and the other 22 miles. In what time will they meet? Ans. $7\frac{1}{2}$ days.

35. A privateer, sailing at the rate of 9 miles in an hour, discovers a merchant vessel 18 miles distant, sailing at the rate of 7 miles in an hour. In what time will the privateer overtake the other vessel? Ans. 9 hours.

36. A woman bought some apples at 3 for a penny, and as many at 2 for a penny, and sold them all again at 5 for twopence, and found that she had lost sixpence. How many of each kind did she buy? Ans. 180.

37. A hare, 40 of her leaps before a hound, takes 4 leaps for the hound's 3, but 2 of the hound's leaps are equal to 3 of the hare's. How many leaps must the hound take before he catch the hare?

$$\frac{3x}{2}-\frac{4x}{3}=40. \qquad \text{Ans. 240 hound's leaps.}$$

38. A son asked his father's age. The father replied,

7 years ago I was 3 times as old as you were; but if we live together 7 years longer, my age will be the double of yours. What were their ages? Ans. 49 and 21.

39. An army being drawn up in a square, there were 79 men over; but in attempting to enlarge each side of the square by one man, there were 80 men too few. Required the number of men? Ans. 6320 men.

40. The paving of a square court, at 8d. per square yard, cost as much as the enclosing of it at 5s. the yard. Required its extent? Ans. 30 yards each side.

41. A person lost $\frac{1}{5}$ of his money by gaming, and then won 4s. Again he lost $\frac{1}{4}$ of what he then had, and afterwards won 3s. The third time he lost $\frac{1}{5}$ of what he then had; and after that, he had remaining $\frac{1}{2}$ of what he began with. How much money had he?

$$\frac{4x}{5}+4-\frac{4x}{20}-1+3-\frac{2x}{10}-2=\frac{x}{2}.$$ Ans. 40s.

42. A cistern can be filled with water by one cock in 12 hours, and by another in 8 hours. In what time will it be filled if both run together? Ans. $4\frac{4}{5}$ hours.

43. The tail of a fish weighed 9 lb., the head weighed as much as the tail and half the body, and the weight of the body was equal to that of the head and tail. What was the weight of the fish? Ans. 72 lb.

44. A gentleman's two horses with the harness cost him £120; the value of the worst horse with the harness was double that of the best horse, and the value of the best horse with the harness was triple that of the worst horse. What was the value of each?

Ans. £50 harness, £40 and £30 horses.

45. A master with his apprentice can perform a piece of work in 8 days, which the master alone could do in 12 days. In what time could the apprentice do it?

$$\frac{x}{8}-\frac{x}{12}=1.$$ Ans. 24 days.

46. Three men can do a piece of work, the first in 50 hours, the second in 60 hours, and the third in 75 hours. In what time will they do it, all working together? Ans. 20 hours.

47. A and B together can do a piece of work in 12 hours, A and C together in 20 hours, and B and C together in 15 hours. In what time will they do it, all working together, and in what time will each do it separately?

$\frac{x}{12}+\frac{x}{20}+\frac{x}{15}=2.$ Ans. Together in 10 hours, A alone in 30 hours, B alone in 20 hours, and C alone in 60 hours.

48. A labourer engages to work 160 days, on condition that he should receive half-a-crown for every day that he wrought, and should forfeit 10d. for every day he was absent from work. At the end of the stipulated time he had nothing to receive nor to pay. How many days did he work?

Ans. Wrought 40 days.

49. To find three numbers, so that the first with $\frac{1}{2}$ of the other two, the second with $\frac{1}{3}$ of the other two, and the third with $\frac{1}{4}$ of the other two, shall each be equal to 34.

Ans. 10, 22, and 26.

50. To find a number consisting of three places, of which the digits have equal differences in their order, and if the number be divided by the sum of its digits, the quotient shall be 48; and if 198 be subtracted from the number, the digits shall be inverted. $100x+10y+z$ the number.

$x+z=2y$, $48\times 3y=$ number, $99x-99z=198$. Ans. 432.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

51. To divide the number 100 into two parts, so that their product shall be 2100. Ans. 70 and 30.

52. To find two numbers, of which the difference shall be 8, and their product 240. Ans. 20 and 12.

53. To find two numbers, of which the difference shall be 12, and the sum of their squares 1424. Ans. 32 and 20.

54. To find two numbers, of which the sum shall be 30, and their product 224. Ans. 16 and 14.

55. To find two numbers, of which the product shall be 108, and the sum of their squares 225. Ans. 12 and 9.

56. A gardener and his lad digged each a square piece of ground, of which the side was as many feet long as the worker was years old. The difference of their ages was 12 years, and the number of square feet digged by both was 1040. Required their ages? Ans. 28 and 16.

57. An oblong pond was surrounded by a terrace-walk 7 yards broad, the pond measured 15000 square yards, and the walk 3696 square yards. Required the length and breadth of the pond?

$$xy=15000, \text{ and } 14x+14y+196=3696.$$

Ans. 150 and 100 yards.

58. To find two numbers of which the sum is 13, and the sum of their cubes 637. Ans. 8 and 5.

59. To find two numbers, of which the product shall be 120, and the product of the greater, increased by 8, multiplied by the less, increased by 5, shall be 300.

Ans. 12 and 10, or 16 and $7\frac{1}{2}$.

60. To divide 125 into two parts, so that the sum of their square roots shall be 15.

$$\sqrt{y}+(125-y)^{\frac{1}{2}}=15. \qquad \text{Ans. } 100 \text{ and } 25.$$

61. A grazier bought a number of sheep for £60, and, reserving 15 to himself, he sold the remainder for £54, and gained 2s. on each of them. How many sheep did he buy, and what did each cost? Ans. 75 sheep at 16s.

62. Sold an ox for £24, and gained as much per cent. as the ox cost. What was paid for him?

$$x+\frac{x^2}{100}=24. \qquad \text{Ans. £}20.$$

63. A person bought some oxen for £80: if he had got 4 oxen more for the same money, each of them would have cost him £1 less. How many did he buy? Ans. 16.

64. A number of bees alighted upon a tree: at the first flight the square root of $\frac{1}{2}$ of them went away, and at the next $\frac{8}{9}$ of them, and then only two bees remained. How many alighted on the tree?

$$\sqrt{\tfrac{1}{2}x}+\frac{8x}{9}+2=x. \qquad \text{Ans. } 72 \text{ bees.}$$

65. A person bought cloth for £33, 15s., which he sold again at £2, 8s. per piece, and gained as much as a piece cost him. Required the number of pieces? Ans. 15 pieces.

66. A and B set out at the same time for a place at the distance of 150 miles. A travels 3 miles an hour faster than B, and arrives at his journey's end $8\frac{1}{3}$ hours before him. At what rate per hour did each person travel?

Ans. A 9 miles, B 6 miles.

67. There are two numbers, of which the product is 120: if 2 be added to the less, and 3 subtracted from the greater, the product of the sum and difference will be also 120. What are the numbers? Ans. 15 and 8.

68. A and B distribute each £1200 among some poor persons: A relieves 40 persons more than B, and B gives £5 a-piece to each person more than A. How many persons were relieved by A and B? Ans. 120 by A, 80 by B.

69. A person bought some sheep for £57, but he lost 8 of them, and then sold the remainder at 8s. a-head profit; and thus he neither gained nor lost by the bargain. How many sheep did he buy? Ans. 38.

70. To divide the number 18 into two factors, so that the sum of their cubes shall be 243. Ans. 6 and 3.

71. There is a number consisting of two digits, the left-hand digit is 3 times the other; and if 12 be subtracted from the

number, the remainder will be the square of the left-hand digit. What is the number? Ans. 93.

72. A, travelling to London, overtook at the 50th milestone a flock of sheep, proceeding at the rate of 3 miles in 2 hours; and 2 hours afterwards met a waggon moving at the rate of 9 miles in 4 hours. B, travelling at the same rate, overtook the sheep at the 45th milestone, and met the waggon 40 minutes before he came to the 31st milestone. Where would B be when A reached London? $x=$ distance between them, $y=$ rate of their travelling per hour, $\frac{10y}{3}-5=x$, $50-2y-\frac{32y^2}{27}+\frac{76y}{9}$ $=31+\frac{2y}{3}-x$. Ans. $x=25$, $y=9$.

OF RATIOS.

Ratio is the relation which one quantity bears to another of a *similar kind* with respect to its magnitude.

The *magnitude* or *value* of a ratio is estimated by stating how often one quantity *contains* or is *contained* in another. Thus, in comparing the number 16 with 2, we observe that it has a certain magnitude with respect to 2 which it contains 8 times; and if we compare 16 with 4 we observe that it has a different relative magnitude, for it contains 4 only 4 times. Hence 16 is less when compared with 4, than it is when compared with 2.

The general method of expressing the ratio which one quantity bears to another is by placing two points between them. Thus

The ratio of 12 to 4 is expressed by $12:4$
. . . . . . of 17 to 9 . . . . by $17:9$
. . . . . . of a to b . . . . by $a:b$.

The first term of a ratio is called the *Antecedent*, and the last term the *Consequent*. The antecedents in the preceding ratios are therefore 12, 17, and a, and the consequents 4, 9, and b. Ratios may also be represented in the form of fractions, by making the antecedents the numerators, and the consequents the denominators: Thus $\frac{12}{4}$, $\frac{17}{9}$, and $\frac{a}{b}$ express the ratios of 12 to 4, of 17 to 9, and of a to b.

A ratio is said to be a ratio of *greater inequality* when the antecedent is greater than the consequent, a ratio of *equality* when it is equal to the consequent, and a ratio of *less inequality* when it is less than the consequent: Thus

The ratio of $8:\ 4$ or $a+b:a$ is a ratio of greater inequality.
. . . . . . of $8:\ 8$ or $a\quad :a$ . . . . . of equality.
. . . . . . of $8:12$ or $a\quad :a+b$ . . . of less inequality.

NOTE. It is evident that a ratio of equality may always be represented by unity.

COMPARISON OF RATIOS.

I. If the terms of a ratio are both multiplied or both divided by the same quantity, the value of the ratio is not altered.

The ratio of $a:b$ is expressed by the fraction $\frac{a}{b}$. Let both terms of this fraction be multiplied by n, and it becomes $\frac{na}{nb}$; now since the value of a fraction is not altered by multiplying both the numerator and denominator by the same quantity $\frac{a}{b}=\frac{na}{nb}$, or the ratio of $a:b$ is the same as the ratio of $na:nb$ where n may be any number either integral or fractional: Thus

The ratio of $16:12$ (divid. by 4) is the same as the ratio of $4:\ 3$.
. . . . . of $5:\ 8$ (mult. by 3) . . . . of $15:24$.
. . . . . of $a^2:ab$ (divid. by a) . . . . of $a:b$.

II. Ratios are compared together by reducing the fractions which represent them to a common denominator.

Thus the ratios of $7:9$ and $10:13$ are represented by the fractions $\frac{7}{9}$ and $\frac{10}{13}$, which are equivalent to $\frac{91}{117}$ and $\frac{90}{117}$; and since $\frac{91}{117}$ is greater than $\frac{90}{117}$, we infer that the ratio of $7:9$ is greater than that of $10:13$.

When the antecedents or consequents are the same in two or more ratios, we may immediately compare those ratios together, by expressing them in a fractional form: Thus since $\frac{17}{5}$ is greater than $\frac{17}{9}$, the ratio of $17:5$ is greater than that of $17:9$; and since $\frac{a}{a+b}$ is less than $\frac{a}{b}$, the ratio of $a:a+b$ is less than that of $a:b$.

III. A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same quantity to both of its terms.

Let $\frac{a}{b}$ represent any ratio, and add n to each of its terms,

then these two ratios will be $\frac{a}{b}$ and $\frac{a+n}{b+n}$, which are equivalent to $\frac{ab+an}{b(b+n)}$ and $\frac{ab+bn}{b(b+n)}$; now if $a > b$, then $\frac{a}{b}$ is a ratio of greater inequality, and $\frac{ab+an}{b(b+n)} > \frac{ab+bn}{b(b+n)} \therefore \frac{a}{b}$ is diminished by adding n to each of its terms; again, if $a < b$, then $\frac{a}{b}$ is a ratio of less inequality, and $\frac{ab+an}{b(b+n)} < \frac{ab+bn}{b(b+n)} \therefore \frac{a}{b}$ is increased by the addition of n to both its terms.

COMPOSITION OF RATIOS.

I. Ratios are compounded by multiplying their antecedents together to form a new antecedent, and their consequents to form a new consequent, and the resulting ratio is called the *sum* of the compounding ratios. Thus
The ratio of $a : b$ is compounded with the ratio of $c : d$ by multiplying the antecedents a and c together for a new antecedent, and the consequents b and d together for a new consequent, and the resulting ratio of $ac : bd$ is the sum of the compounding ratios $a : b$ and $c : d$.

If the ratios $4 : 7$, $6 : 11$, and $7 : 9$, are compounded together, the resulting ratio is $4 \times 6 \times 7 : 7 \times 11 \times 9$ or $168 : 693$, which, reduced to its lowest terms by dividing both terms by 21, becomes the ratio of $8 : 33$.

II. When any ratio $a : b$ is compounded with itself twice, thrice, or any number of times, denoted by n, then the resulting ratios are $a^2 : b^2$, $a^3 : b^3$, $a^n : b^n$, or twice, thrice, and n times the ratio of $a : b$.

The ratios $a^2 : b^2$, $a^3 : b^3$, $a^4 : b^4$, &c. are also called the *duplicate, triplicate, quadruplicate,* &c. ratios of the primitive.

As the indices or exponents 2, 3, and n, express the number of times the ratio of $a : b$ is compounded with itself, they are called the measures of these ratios.

III. Since the index may be any quantity either integral or fractional, let it be a fraction, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{m}$, &c., then

The ratio of $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is $\frac{1}{2}$ the ratio of $a : b$.
. . . . $a^{\frac{1}{3}} : b^{\frac{1}{3}}$ is $\frac{1}{3}$ . . . . . of $a : b$.
. . . . $a^{\frac{1}{4}} : b^{\frac{1}{4}}$ is $\frac{1}{4}$ . . . . . of $a : b$.
. . . . $a^{\frac{1}{m}} : b^{\frac{1}{m}}$ is $\frac{1}{m}$th . . . . of $a : b$.

The ratios of $a^{\frac{1}{2}}:b^{\frac{1}{2}}$, $a^{\frac{1}{3}}:b^{\frac{1}{3}}$, $a^{\frac{1}{4}}:b^{\frac{1}{4}}$, &c. are also called the *subduplicate, subtriplicate, subquadruplicate*, &c. ratios of the primitive.

IV. The sum of any number of ratios, of which the consequent of the preceding ratio is the antecedent of the succeeding one, is the ratio of the first antecedent to the last consequent.

Let the ratios be $a:b$, $b:c$, $c:d$, $d:e$, $e:f$, &c., then the resulting ratio is $a\times b\times c\times d\times e:b\times c\times d\times e\times f$, or the ratio of $abcde:bcdef$, which, reduced to its least terms, by dividing both its terms by $bcde$, becomes the ratio of $a:f$, or first antecedent : last consequent.

V. Any ratio compounded with a ratio of greater inequality is increased, and compounded with a ratio of less inequality is diminished.

Let $a+b:a$ represent the ratio of greater inequality,
and $a:a+b$ . . . . of less inequality.

Then the ratio of $a+b:a$, compounded with that of $c:d$, gives $ac+bc:ad$, which is evidently greater than the ratio of $c:d$; and the ratio of $a:a+b$, compounded with that of $c:d$, gives $ac:ad+bd$, which is evidently less than the ratio of $c:d$. Hence the ratio of $c:d$ is increased by compounding it with the ratio of $a+b:a$, and diminished by compounding it with the ratio of $a:a+b$.

APPROXIMATION OF RATIOS.

The ratio of the powers or roots of two quantities, whose difference is small with respect to themselves, is found very nearly by multiplying that difference by the index or exponent of the power or root.

Let $x+z$ and x be two quantities, whose difference is z; then $(x+z)^n=x^n+nx^{n-1}z+\frac{n(n-1)}{2}x^{n-2}z^2+\frac{n(n-1)(n-2)}{2\times 3}\times x^{n-3}z^3+$&c.; hence the ratio of $(x+z)^n:x^n$ is that of $x^n+nx^{n-1}z+\frac{n(n-1)}{2}x^{n-2}z^2+\frac{n(n-1)(n-2)}{2\times 3}x^{n-3}z^3+$ &c. : x^n, and dividing by x^{n-1} this ratio becomes that of $x+nz+\frac{n(n-1)z^2}{2x}+\frac{n(n-1)(n-2)z^3}{2\times 3x^2}+$ &c. : x.

Now if z is very small with respect to x, the fractions $\frac{z^2}{x}$ and $\frac{z^3}{x^2}$ must also be very small; and when n is not very large, the fractional terms of the series which forms the antecedent

will also be very small with respect to the integral part $x+nz$; hence the ratio of $x+nz:x$ will be a near approximation to the ratio of $(x+z)^n:x^n$, which is the rule.

Let it be required to approximate to the ratio of $1000^3:999^3$.

Here $x=999$, $z=1$, and $n=3$ $\therefore$ the ratio of $x+nz:x$ is that of $1002:999$, which is true to *three* places of decimals very nearly; since $1000^3:999^3::1002:998\cdot997$.

Let it be required to approximate to the ratio of $\sqrt[3]{480}:\sqrt[3]{477}$.

Here $x=477$, $z=3$, and $n=\frac{1}{3}$ $\therefore$ the ratio of $x+nz:x$ is that of $478:477$, which is true to *three* places of decimals very nearly; since $\sqrt[3]{480}:\sqrt[3]{477}::478:477\cdot002$.

EXERCISES.

1. Reduce the ratio of $375:420$ to its lowest terms. Ans. $25:28$.

2. Reduce the ratio of $a^4-a^3b+a^2:a^2$ to its lowest terms. Ans. $a^2-ab+1:1$.

3. Show that the ratio of $18:13$ is the same as that of $162:117$.

4. Show that the ratio of $17:21$ is less than that of $153:168$.

5. Which is the greater ratio, that of $a+6:\frac{1}{3}a+9$, or that of $a+9:\frac{1}{3}a+10$? Ans. That of $a+9:\frac{1}{3}a+10$.

6. Is the ratio of $13:18$ increased or diminished by adding 7 to each of its terms? Ans. Increased.

7. Is the ratio $a:a-b$ increased or diminished by adding c to each of its terms? Ans. Diminished.

8. Approximate to the ratios of $740^2:738^2$ and of $\sqrt{740}:\sqrt{738}$, and show to how many places of decimals the approximation is true. Ans. $742:738$ and $739:738$, the former being true to *three* and the latter to *four* places very nearly.

9. What is the sum of the ratios of $8:11$, $9:13$, $5:7$, and $21:23$? Ans. $1080:3289$.

10. What is the sum of the ratios of $a^3-x^3:a^2$, $a^2+x^2:b^2$, $a-x:b$, and $b:a-x$?

Ans. $a^5+a^3x^2-a^2x^3-x^5:a^2b^2$.

11. Is the sum of the ratios of $5a-1:4a+1$, $3a+6:2a+4$, and $2a+1:3a-1$, a ratio of greater or less inequality? Ans. A ratio of greater inequality.

12. Show that the sum of the ratios of $a+b:x$, $a-b:y$, and $y:\dfrac{a^2-b^2}{x}$, is a ratio of equality.

13. Find in its lowest terms the sum of the duplicate ratio of $8:11$, the triplicate ratio of $5:4$, and the ratio of $22:15$. Ans. $50:33$.

14. Tell extempore the sum of the ratios of $4 : 3^2$, $3^2 : 5^3$, $5^3 : 7^2$, and $7^2 : 9$.

15. Of what two simple ratios is the ratio of $9 : 24$ compounded? Ans. Of the ratios of $1 : 2$ and $3 : 4$.

OF PROPORTION.

PROPORTION consists in the *equality of ratios*.

Thus if the ratio of $a : b$ is equal to that of $c : d$, or $\frac{a}{b} = \frac{c}{d}$; then a, b, c, d, are said to be proportionals. The numbers 3, 12, 4, 16, are proportionals for $\frac{3}{12} = \frac{1}{4}$, and $\frac{4}{16} = \frac{1}{4}$.

This equality of ratios is expressed by writing the four quantities, thus $a : b :: c : d$, and read a is to b as c is to d. In Algebraic investigations the quantities are generally expressed like fractions, thus $\frac{a}{b} = \frac{c}{d}$.

In the proportion $a : b :: c : d$ or $\frac{a}{b} = \frac{c}{d}$, a and d are the *extremes*, and b and c the *means*. The first term is likewise called the first antecedent, the second term the first consequent, the third term the second antecedent, and the fourth term the second consequent.

If, in a series of proportional quantities each consequent be identical with the next antecedent, these quantities are said to be in *continued* proportion: Thus $a : b :: b : c :: c : d :: d : e :: e : f$, &c.; the quantities a, b, c, d, e, f, &c. are said to be in *continued* proportion; when the second and third terms of a proportion are identical, as in the proportion $a : b :: b : c$; then b is said to be a *mean proportional* between the extremes a and c, and c is called a *third proportional* to a and b.

PROP. I. If four quantities are proportional, the product of the extremes is equal to the product of the means, and conversely.

Let $a : b :: c : d$ or $\frac{a}{b} = \frac{c}{d}$. Multiplying both by bd, we obtain $ad = bc$.

Conversely. If the product of any two quantities is equal to the product of any other two, these four quantities constitute a proportion, the factors of either of the products being made the extremes, and the factors of the other the means.

Let $ad = bc$. Dividing both by bd, we obtain $\frac{a}{b} = \frac{c}{d}$ or $\frac{c}{d} = \frac{a}{b}$; whence $a : b :: c : d$ or $c : d :: a : b$.

Prop. II. If three quantities are in continued proportion, the product of the extremes is equal to the square of the mean, and conversely.

Let $a:b::b:c$; then $a\times c=b\times b$ or $ac=b^2$.

Conversely. If the product of any two quantities is equal to the square of a third, the third is a mean proportional between the other two.

Let $ac=b^2$, and, dividing both by bc, we obtain $\frac{a}{b}=\frac{b}{c}$ or $a:b::b:c$.

Prop. III. Of four proportionals, any three being given, the fourth may be found.

Let $a:b::c:d$; then $ad=bc$.

Hence $a=\frac{bc}{d}$ $\therefore$ if b, c, d, are known, a is also known.

. . . . $b=\frac{ad}{c}$ . . $a, d, c,$ . . . . . . b . . . .

. . . . $c=\frac{ad}{b}$ . . $a, d, b,$ . . . . . . c . . . .

. . . . $d=\frac{bc}{a}$ . . $a, b, c,$ . . . . . . d . . . .

Hence of three proportionals, any two being given, the third may be found; for $ad=b^2$ $\therefore$ $b=\sqrt{ad}$, $a=\frac{b^2}{d}$, and $d=\frac{b^2}{a}$.

Prop. IV. Quantities which have the same ratio to the same quantity are equal to one another, and conversely.

Let $a:b::c:b$; then $\frac{a}{b}=\frac{c}{b}$, and, multiplying each by b, we obtain $a=c$.

Conversely. Quantities which are equal to one another have the same ratio to the same quantity.

Let $a=c$, and let b be a third quantity; then, dividing both by b, we obtain $\frac{a}{b}=\frac{c}{b}$ $\therefore$ $a:b::c:b$.

Prop. V. Ratios that are equal to the same ratio are equal to one another.

Let $a:b::e:f$, and $c:d::e:f$; then also $a:b::c:d$.

Since $\frac{a}{b}=\frac{e}{f}$ and $\frac{c}{d}=\frac{e}{f}$ $\therefore$ $\frac{a}{b}=\frac{c}{d}$ or $a:b::c:d$.

Prop. VI. If four quantities are proportionals, they will also be proportionals *invertendo*, that is, the second will have the same ratio to the first that the fourth has to the third.

Let $a:b::c:d$; then also $b:a::d:c$.

Since (Prop. I.) $bc=ad$, and, dividing by ac, we get $\frac{b}{a}=\frac{d}{c}$; hence $b:a::d:c$.

Prop. VII. If four quantities are proportionals, they will also be proportionals *alternando*, or the first will have the same ratio to the third that the second has to the fourth.

Let $a:b::c:d$; then also $a:c::b:d$; since $\frac{a}{b}=\frac{c}{d}$, multiply each by $\frac{b}{c}$, and we obtain $\frac{a}{c}=\frac{b}{d}$ $\therefore a:c::b:d$.

Prop. VIII. If four quantities are proportionals, they will also be proportionals *componendo*, or the sum of the first and second will have the same ratio to the second that the sum of the third and fourth has to the fourth.

Let $a:b::c:d$; then also $a+b:b::c+d:d$; since $\frac{a}{b}=\frac{c}{d}$, add 1 to each of these, and we obtain $\frac{a}{b}+1=\frac{c}{d}+1$ or $\frac{a+b}{b}=\frac{c+d}{d}$ $\therefore a+b:b::c+d:d$.

Prop. IX. If four quantities are proportionals, they will also be proportionals *dividendo*, or the difference between the first and second will have the same ratio to the second that the difference between the third and fourth has to the fourth.

Let $a:b::c:d$; then also $a-b:b::c-d:d$; since $\frac{a}{b}=\frac{c}{d}$, subtract 1 from each of these, and we obtain $\frac{a}{b}-1=\frac{c}{d}-1$ or $\frac{a-b}{b}=\frac{c-d}{d}$ $\therefore a-b:b::c-d:d$.

Prop. X. If four quantities are proportionals, they will also be proportionals *convertendo*, or the first will have the same ratio to the sum or difference of the first and second that the third has to the sum or difference of the third and fourth.

Let $a:b::c:d$; then also $a:a\pm b::c:c\pm d$; since $\frac{a}{b}=\frac{c}{d}$, and by Prop. VIII. and IX. $\frac{a\pm b}{b}=\frac{c\pm d}{d}$, invert these fractions, and we have $\frac{b}{a\pm b}=\frac{d}{c\pm d}$; and, multiplying the

one by $\frac{a}{b}$ and the other by $\frac{c}{d}$, we obtain $\frac{b}{a \pm b} \times \frac{a}{b} = \frac{d}{c \pm d} \times \frac{c}{d}$ or $\frac{a}{a \pm b} = \frac{c}{c \pm d}$ $\therefore a : a \pm b :: c : c \pm d$.

PROP. XI. If four quantities are proportionals, the sum of the first and second has the same ratio to their difference that the sum of the third and fourth has to their difference.

Let $a : b :: c : d$; then also $a + b : a - b :: c + d : c - d$.
For taking VIII. and IX. *alternando*, $a + b : c + d :: b : d$ and $a - b : c - d :: b : d$; hence (V.), $a + b : c + d :: a - b : c - d$ (and *alternando*), $a + b : a - b :: c + d : c - d$.

PROP. XII. In any number of proportionals any antecedent has the same ratio to its consequent that the sum of all the antecedents has to the sum of all the consequents.

Let $a : b :: c : d :: e : f :: g : h$, &c.; then also $a : b :: a + c + e + g +$ &c. $: b + d + f + h +$ &c.
Since $ab = ba$, $ad = bc$, $af = be$, $ah = bg$, &c., we have $a(b + d + f + h +$ &c.$) = b(a + c + e + g +$ &c.$)$; whence $\frac{a}{b} = \frac{a + c + e + g + \text{\&c.}}{b + d + f + h + \text{\&c.}}$ $\therefore a : b :: a + c + e + g +$ &c. $: b + d + f + h +$ &c. In like manner it may be shown that $c : d :: a + c + e + g +$ &c. $: b + d + f + h +$ &c.

PROP. XIII. In two or more ranks of proportionals the products of the corresponding terms are also proportionals.

$$\left.\begin{array}{l} \text{Let } a : b :: c : d \\ \quad\; e : f :: g : h \\ \quad\; i : k :: l : m \end{array}\right\} \text{Then also } aei : bfk :: cgl : dhm.$$

Since $\frac{a}{b} = \frac{c}{d}$, $\frac{e}{f} = \frac{g}{h}$, $\frac{i}{k} = \frac{l}{m}$; then $\frac{aei}{bfk} = \frac{cgl}{dhm}$ $\therefore aei : bfk :: cgl : dhm$.

PROP. XIV. If there are any number of quantities more than two, and as many others, which, taken two and two in order, are proportionals; then *ex æquo* are the extreme terms in the former series, proportional to the extreme terms in the latter.

Let a, b, c, d, be any number of quantities,
and e, f, g, h, as many others.

$$\left.\begin{array}{l} \text{Let } a : b :: e : f \\ \quad\; b : c :: f : g \\ \quad\; c : d :: g : h \end{array}\right\} \text{Then also } a : d :: e : h.$$

Since $\frac{a}{b}=\frac{e}{f}$, $\frac{b}{c}=\frac{f}{g}$, and $\frac{c}{d}=\frac{g}{h}$, we obtain, by multiplying the alternate fractions together, $\frac{abc}{bcd}=\frac{efg}{fgh}$ or $\frac{a}{d}=\frac{e}{h}$ $\therefore a:d::e:h$.

Prop. XV. If there are any number of quantities more than two, and as many others, which, taken two and two in a cross order, are proportionals; then *ex æquo inversely* are the extreme terms in the first rank, proportional to the extreme terms in the second.

Let $a, b, c, d,$ be any number of terms,
and $e, f, g, h,$ as many others;

and let $\left.\begin{array}{l} a:b::g:h \\ b:c::f:g \\ c:d::e:f \end{array}\right\}$ Then also $a:d::e:h$.

Since $\frac{a}{b}=\frac{g}{h}$, $\frac{b}{c}=\frac{f}{g}$, and $\frac{c}{d}=\frac{e}{f}$; by multiplying the alternate fractions together, we obtain $\frac{abc}{bcd}=\frac{gfe}{hgf}$ or $\frac{a}{d}=\frac{e}{h}$ $\therefore a:d::e:h$.

Prop. XVI. When four quantities are proportionals, if the first and second are multiplied or divided by any quantity, and also the third and fourth by the same or any other quantity, the resulting quantities will be proportionals.

Let $a:b::c:d$; then also $ma:mb::nc:nd$.

Since $\frac{a}{b}=\frac{c}{d}$, multiply both terms of the first by m, and both terms of the last by n, and we obtain $\frac{ma}{mb}=\frac{nc}{nd}$ $\therefore ma:mb::nc:nd$, where m and n may be any quantities either integral or fractional.

Prop. XVII. When four quantities are proportionals, if the first and third are multiplied or divided by the same quantity, and also the second and fourth by the same quantity, the resulting quantities will be proportionals.

Let $a:b::c:d$; then also $ma:nb::mc:nd$.

Since $\frac{a}{b}=\frac{c}{d}$, multiply both these by $\frac{m}{n}$, and we obtain $\frac{ma}{nb}=\frac{mc}{nd}$ $\therefore ma:nb::mc:nd$, where m and n may be any quantities either integral or fractional.

Prop. XVIII. If four quantities are proportionals, the like powers or roots of these quantities are also proportionals.

Let $a : b :: c : d$; then also $a^m : b^m :: c^m : d^m$.

Since $\frac{a}{b} = \frac{c}{d}$ raise each of these fractions to the power expressed by m; then $\left(\frac{a}{b}\right)^m = \left(\frac{c}{d}\right)^m$ or $\frac{a^m}{b^m} = \frac{c^m}{d^m}$ $\therefore$ $a^m : b^m :: c^m : d^m$, where m may be any quantity either integral or fractional.

Prop. XIX. Of any number of quantities in continued proportion, the first has to the third the *duplicate* ratio, to the fourth the *triplicate* ratio, to the fifth the *quadruplicate* ratio, &c. of that which it has to the second, or of that which the second has to the third, &c.

Let $a : b :: b : c :: c : d :: d : e :: e : f ::$ &c. &c.
Then $a : c :: a^2 : b^2$ or in the duplicate ratio of $a : b$.
$a : d :: a^3 : b^3$ . . . . . triplicate ratio of $a : b$.
$a : e :: a^4 : b^4$ . . . . . quadruplicate ratio of $a : b$.
&c. &c. &c. &c.

1st, $a : b :: b : c$, or by Prop. XVIII., $a^2 : b^2 :: b^2 : c^2$; but by Prop. II., $b^2 = ac$ $\therefore$ $a^2 : b^2 :: ac : c^2$ or $a^2 : b^2 :: a : c$; hence $a : c :: a^2 : b^2$; also $a^2 : ac :: b^2 : c^2$ $\therefore$ $a : c :: b^2 : c^2$.

2d, $a : c :: a^2 : b^2$;
but $c : d :: a \; : b$
$\therefore$ $a : d :: a^3 : b^3 :: b^3 : c^3 :: c^3 : d^3$.

3d, $a : d :: a^3 : b^3$.
and $d : e :: a \; : b$.
$\therefore$ $a : e :: a^4 : b^4 :: b^4 : c^4 :: c^4 : d^4 :: d^4 : e^4$.

EXERCISES.

1. There are two numbers which are to each other as $5 : 4$, and if 5 is added to the greater, and 1 subtracted from the less, the sum will be to the remainder as $5 : 3$. What are the numbers? Ans. 20 and 16.

2. Divide the number 120 into two such parts that their product shall be to the difference of their squares as $2 : 3$. Ans. 80 and 40.

3. The number 45 is divided into two parts, which are to each other in the triplicate ratio of $4 : 2$. Find a mean proportional between them. Ans. 14·14213.

4. The product of two numbers is 48, and the difference of their cubes is to the cube of their difference as $37 : 1$. What are the numbers? Ans. 8 and 6.

5. Divide the number 100 into two such parts that 6 times their product shall be to the sum of their squares as $24 : 17$. Ans. 80 and 20.

6. There are two numbers whose product is 15, and the difference of their squares is to the square of their difference as $4:1$. What are the numbers? Ans. 5 and 3.

7. Let $x^2 : y^2 :: 49 : 36$ and $2x - y : x + 6$, in a ratio compounded of the ratios of $2^5 : 2^2$ and $2 : 5$. Required the values of x and y. Ans. $x = 14$, and $y = 12$.

8. There are two numbers in the triplicate ratio of $4:1$ whose mean proportional is 32. What are the numbers?
Ans. 256 and 4.

9. If $dx = cy$ and $x : y$ in the triplicate ratio of $a : b$; show that the ratio of $a : b$ is that of $\sqrt[3]{c+x} : \sqrt[3]{d+y}$.

OF VARIABLE QUANTITIES.

Quantities which alter their values are called Variable Quantities, and they are often so related to one another, that when one of them is increased the others are increased or diminished according to a constant rule.

Thus if a body moves uniformly, the space it describes increases in the same ratio with the time.

Let S and s be two spaces, T and t the times in which they are described, then $S : T :: s : t$ or $S : s :: T : t$ where S is said to *vary directly*, as T, and this relation is written $S \propto T$.

If the relation between S and T is such, that whilst S by increasing becomes s, and T by diminishing becomes t, in such a manner that in all cases $S : s :: t : T$ or $S : s :: \frac{1}{T} : \frac{1}{t}$; then S is said to *vary inversely*, as T, and is expressed $S \propto \frac{1}{T}$.

If three quantities, S, T, V, are so related to one another, that when S is increased to s, $T \times V$ is also increased to $t \times v$, so that in all cases $S : s :: TV : tv$; then S is said to *vary*, as T and V *jointly*, and is written $S \propto TV$.

If the three variable quantities are so related to one another, that when V is increased to v, S is also increased to s, and T diminished to t, so that in all cases $V : v$ in the ratio compounded of the ratios of $S : s$ and $\frac{1}{T} : \frac{1}{t}$ or $V : v :: \frac{S}{T} : \frac{s}{t}$; then V is said to *vary directly*, as S, and *inversely*, as T, and is written $V \propto \frac{S}{T}$. In this case, if S is constant, $V \propto \frac{1}{T}$ or V is said to vary *inversely*, as T.

These are called *general proportions;* and if the values

of the variable quantities can be determined at a given period of their increase or decrease, they may be reduced to determined proportions.

PROP. I. If $S \propto T$, and $T \propto V$, then $S \propto V$. For $S : s :: T : t$ and $T : t :: V : v \therefore S : s :: V : v$; hence $S \propto V$.

PROP. II. If $S \propto T$, and $T \propto \frac{1}{V}$, then $S \propto \frac{1}{V}$. For $S : s :: T : t$ and $T : t :: v : V :: \frac{1}{V} : \frac{1}{v} \therefore S : s :: \frac{1}{V} : \frac{1}{v}$; hence $S \propto \frac{1}{V}$.

PROP. III. If $S \propto V$ and $T \propto V$, then $S \pm T \propto V$. For $S : s :: V : v$ and $T : t :: V : v \therefore S : s :: T : t$, or alternando, $S : T :: s : t$, componendo, $S \pm T : T :: s \pm t : t$, and alternando, $S \pm T : s \pm t : T : t$, but $T : t :: V : v \therefore S \pm T : s \pm t :: V : v$; hence $S \pm T \propto V$.

PROP. IV. If $S \propto V$, and $T \propto V$, then $V \propto \sqrt{ST}$. For $S : s :: V : v$ and $T : t :: V : v \therefore ST : st :: V^2 : v^2$; hence $\sqrt{ST} : \sqrt{st} :: V : v$ or $V \propto \sqrt{ST}$.

PROP. V. If $S \propto T$, and $V \propto X$, then $SV \propto TX$. For $S : s :: T : t$ and $V : v :: X : x \therefore SV : sv :: TX : tx$; hence $SV \propto TX$.

PROP. VI. If $S \propto T$, then $S \propto nT$, where n may be any number either integral or fractional. For $S : s :: T : t \therefore$ multiplying the last ratio by n, we get $S : s :: nT : nt$; hence $S \propto nT$.

Cor. If $S \propto T$, then $S = T$ multiplied by some constant quantity. For $S : s :: T : t$, or alternando, $S : T :: s : t$; hence in every state of the quantities the ratio of $S : T$ is the same. Let it be that of $n : 1$, then $S : T :: n : 1 \therefore S = nT$ or $n = \frac{S}{T}$; hence the value of n will be known, if the corresponding values of S and T at any period of their variation be known.

PROP. VII. If $S \propto T$, then $S^n \propto T^n$, where n may be any number either integral or fractional. For $S : s :: T : t \therefore$ by Prop. XVIII. of Proportion, $S^n : s^n :: T^n : t^n$; hence $S^n \propto T^n$.

PROP. VIII. If $(V+T)^2 \propto (V-T)^2$, then $V^2+T^2 \propto VT$. For $(V+T)^2 : (v+t)^2 :: (V-T)^2 : (v-t)^2$ or $(V+T)^2 : (V-T)^2 :: (v+t)^2 : (v-t)^2$, and by Prop. XI. of Proportion, $2V^2+2T^2 : 4VT :: 2v^2+2t^2 : 4vt$, di-

viding by 2, we get $V^2+T^2:2VT::v^2+t^2:2vt$ or $V^2+T^2:v^2+t^2::2VT:2vt::VT:vt$; hence $V^2+T^2 \propto VT$.

Prop. IX. If $V \propto T$, then $SV \propto ST$, and $\frac{V}{S} \propto \frac{T}{S}$. For $V:v::T:t$ and $S:s::S:s$ $\therefore$ $SV:sv::ST:st$, also $\frac{V}{S}:\frac{v}{s}::\frac{T}{S}:\frac{t}{s}$; hence $SV \propto ST$ and $\frac{V}{S} \propto \frac{T}{S}$.

Prop. X. If there are two ranks of quantities, S, T, V, &c. and X, Y, Z, &c. related in such a manner, that $S \propto X$, $T \propto Y$, $V \propto Z$, &c.; then will STV, &c. $\propto$ XYZ, &c. For $S:s::X:x$, $T:t::Y:y$, $V:v::Z:z$, &c. $\therefore$ by Prop. XIII. of Proportion, STV, &c. : stv, &c. : : XYZ, &c. : xyz, &c.; hence STV, &c. $\propto$ XYZ, &c.

Prop. XI. If S depends upon T, V, X, in such a manner, that $S \propto T$, when V and X are constant; $S \propto V$, when T and X are constant; and $S \propto X$, when T and V are constant; then $S \propto TVX$, when they all vary. For let the respective values of S be s, a, b; then when all the quantities vary, we have $\left.\begin{array}{l} S:s::T:t \\ s:a::V:v \\ a:b::X:x \end{array}\right\}$ Hence, by composition of ratios, $S:b::TVX:tvx$ or $S \propto TVX$, which must be true, whatever be the number of quantities.

LITERAL ANALYSIS.

When the known quantities are expressed in numbers, these numbers disappear during the progress of the operation, and the answer, when obtained, does not exhibit the process by which it has been deduced from the assumed data. This mode, though generally adopted in the solution of practical exercises, does not exhibit sufficiently the true difference between arithmetic and algebra, but rather confounds them. The essential character of algebra, taken in its most extensive meaning, is, that the results of its operations do not give the particular values of the quantity or quantities sought; they only represent the operations which ought to be made upon the given quantities, for obtaining the values of those sought, according to the conditions of the problem; so that the principal object of algebra is the investigation of theorems and the exhibition of rules for the arithmetical or geometrical solution of problems. For accomplishing these purposes, it is necessary to represent

the known quantities by letters, as well as the unknown ones. The former are represented by the first letters of the alphabet, a, b, c, &c. and the unknown ones by the last letters, x, y, z, &c. The question is translated into equations, and these equations are resolved by the preceding rules; and then the values of the unknown quantities will be expressed in a general way from their relations to those which are given in the question. Consequently, if this general expression be transferred from algebraical characters into common language, it will give a general rule for the solution of all questions of the same kind. But the expressions will answer the same purpose as accurately in algebraical characters, and then they are called Theorems, or Formulæ.

1. Given the sum s, and the difference d, of two quantities x and y; to find the quantities. $x+y=s$, and $x-y=d$: by adding these equations we get $2x=s+d$, whence $x=\frac{s+d}{2}$; and by subtracting the equations we get $2y=s-d$, and $y=\frac{s-d}{2}$. These values, expressed in common language, give the following rules, viz.

To find the greater, add the difference to the sum, and divide by 2.

To find the less, subtract the difference from the sum, and divide by 2.

2. Given the sum s, of two quantities x and y, and the difference of their squares D; to find the quantities. $x+y=s$, and $x^2-y^2=D$; and dividing the latter by the former, we get $x-y=\frac{D}{s}$; whence, as before, $x=\frac{s}{2}+\frac{D}{2s}$ and $y=\frac{s}{2}-\frac{D}{2s}$, or $x=\frac{s^2+D}{2s}$ and $y=\frac{s^2-D}{2s}$.

3. As exercises, the student may investigate the following, viz. Of two quantities, their sum, difference, product, quotient, sum and difference of their squares, any two being given; to find all the rest. The operations will be similar to those used in the two last questions; and the results, except for the sum and difference of their squares, are given in the following Table, in which x and y are the quantities, $s=$ their sum, $d=$ their difference, $p=$ their product, $q=$ their quotient, $Z=$ the sum of their squares, and $D=$ the difference of their squares.

TABLE.

Given.	Greater $= x$.	Less $= y$.	Sum $= s$.	Difference $= d$.	Product $= p$.	Quotient $= q$.
s and d	$\frac{s+d}{2}$	$\frac{s-d}{2}$			$\frac{s^2-d^2}{4}$	$\frac{s+d}{s-d}$
s and p	$\frac{s+(s^2-4p)^{\frac{1}{2}}}{2}$	$\frac{s-(s^2-4p)^{\frac{1}{2}}}{2}$		$(s^2-4p)^{\frac{1}{2}}$		$\frac{s+(s^2-4p)^{\frac{1}{2}}}{s-(s^2-4p)^{\frac{1}{2}}}$
s and q	$\frac{sq}{q+1}$	$\frac{s}{q+1}$		$\frac{q-1}{q+1}s$	$\frac{s^2q}{(q+1)^2}$	
d and p	$\frac{d+(d^2+4p)^{\frac{1}{2}}}{2}$	$\frac{d-(d^2+4p)^{\frac{1}{2}}}{2}$	$(d^2+4p)^{\frac{1}{2}}$			$\frac{d+(d^2+4p)^{\frac{1}{2}}}{d-(d^2+4p)^{\frac{1}{2}}}$
d and q	$\frac{dq}{q-1}$	$\frac{d}{q-1}$	$\frac{q+1}{q-1}\times d$		$\frac{qd^2}{(q-1)^2}$	
p and q	$(pq)^{\frac{1}{2}}$	$\sqrt{\frac{p}{q}}$	$(q+1)\sqrt{\frac{p}{q}}$	$(q-1)\sqrt{\frac{p}{q}}$		
d and D	$\frac{d^2+D}{2d}$	$\frac{D-d^2}{2d}$	$\frac{D}{d}$		$\frac{D^2-d^4}{4d^2}$	$\frac{D+d^2}{D-d^2}$
Z and D	$\left(\frac{Z+D}{2}\right)^{\frac{1}{2}}$	$\left(\frac{Z-D}{2}\right)^{\frac{1}{2}}$	$(Z+\sqrt{Z^2-D^2})^{\frac{1}{2}}$	$(Z-\sqrt{Z^2-D^2})^{\frac{1}{2}}$	$\frac{\sqrt{Z^2-D^2}}{2}$	$\frac{\sqrt{Z^2-D^2}}{Z-D}$

The use of this Table is plain. Suppose the sum of two numbers to be 277, and their difference to be 115; then the greater number is $\left(\frac{s+d}{2}\right)=\left(\frac{277+115}{2}\right)=\frac{392}{2}=196$.

Suppose again the difference of two numbers to be 10, and their product 119.

The greater number is $\frac{d+(d^2+4p)^{\frac{1}{2}}}{2}=\frac{10+(100+476)^{\frac{1}{2}}}{2}$ $=\frac{10+\sqrt{576}}{2}=\frac{10+24}{2}=17$.

Suppose the sum of their squares to be 250, and the difference of their squares to be 88.

The greater number is $\left(\frac{Z+D}{2}\right)^{\frac{1}{2}}=\left(\frac{250+88}{2}\right)^{\frac{1}{2}}=\sqrt{169}$ $=13$.

The less is $\left(\frac{Z-D}{2}\right)^{\frac{1}{2}}=\left(\frac{250-88}{2}\right)^{\frac{1}{2}}=\sqrt{81}=9$.

4. Given the sum s, of the products of two quantities, by known multipliers m and n, and also the sum of their products c, by other known multipliers p and q, to find the quantities.

Here $mx+ny=s$, and $px+qy=c$; multiplying the former equation by p, and the latter by m, they become $pmx+pny=ps$, and $mpx+mqy=mc$; subtracting, we get $npy-mqy=ps-mc$; and dividing by $np-mq$, we obtain $y=\frac{ps-mc}{np-mq}$; in the same way we find $x=\frac{qs-nc}{mq-np}$.

5. Given the sum s of the quotients of two quantities by known divisors m and n, and also the sum c, of their quotients by other known divisors p and q; to find the quantities.

Here $\frac{x}{m}+\frac{y}{n}=s$, and $\frac{x}{p}+\frac{y}{q}=c$, whence $nx+my=mns$, and $qx+py=pqc$; which, resolved as the last, give $x=\frac{pm(ns-qc)}{pn-qm}$, and $y=\frac{nq(ms-pc)}{qm-pn}$

6. Given the values m and n, of two ingredients; to find the quantities which must be taken of each, to form a given quantity a, of a compound of a given value e.

Here $x+y=a$, and $mx+ny=ae$.

Ans. $x=a\frac{e-n}{m-n}$, and $y=a\frac{e-m}{n-m}$.

7. Given the times m and n, in which two agents could

produce the same effect separately; to find the time in which they could do it jointly.

Here $\frac{x}{m}+\frac{x}{n}=1$. Ans. $x=\frac{mn}{m+n}$.

8. Given the times m, n, and r, in which three agents can perform the same work separately; to find the time in which they can do it jointly.

Here $\frac{x}{m}+\frac{x}{n}+\frac{x}{r}=1$. Ans. $x=\frac{mnr}{mn+mr+nr}$.

9. Given the times m, n, and r, in which every two of three agents can perform the same work; to find the time x, in which they can do it jointly, and also the times y, z, and v, in which each of them can do it separately.

Here $\frac{x}{m}+\frac{x}{n}+\frac{x}{r}=2$. Ans. $x=\frac{2mnr}{mn+mr+nr}$, $y=\frac{2mnr}{(m+n)r-mn}$,

$z=\frac{2mnr}{(m+r)n-mr}$, and $v=\frac{2mnr}{(n+r)m-nr}$.

10. Given the specific gravities m and n, of two ingredients, and the quantity a, of the mixture, with its specific gravity r; to find the quantities of the ingredients.

Ans. $x=\frac{ma(r-n)}{r(m-n)}$, and $y=\frac{na(m-r)}{r(m-n)}$.

11. Given the first distance d, of two moving bodies, and their velocities m and n; to find the time of their conjunction.

Ans. $x=\frac{d}{n\pm m}$, where the upper sign must be used when they move in opposite directions, and the under when they move in the same direction.

12. Given the sum $2s$, of two numbers, and also the sum of their squares, of their cubes, of their fourth, or of their fifth powers, &c.; to find the numbers.

NOTE. If their difference be $2x$, the numbers will be $s+x$ and $s-x$; and then the sum of their squares will be $2s^2+2x^2$, the sum of their cubes $2s^3+6sx^2$, the sum of their fourth powers $2s^4+12s^2x^2+2x^4$, and the sum of their fifth powers $2s^5+20s^3x^2+10sx^4$, all of which are of the quadratic or simple form, and may be resolved as before; but the sums of the higher powers exceed the quadratic.

Let $z=$ sum of their squares, $c=$ sum of their cubes, $q=$ sum of their fourth powers, and $p=$ sum of their fifth powers;

then $x=\left(\frac{z-2s^2}{2}\right)^{\frac{1}{2}} \quad =\left(\frac{c-2s^3}{6s}\right)^{\frac{1}{2}}$

$=\left(-3s^2\pm\sqrt{\tfrac{1}{2}q+8s^4}\right)^{\frac{1}{2}} \quad =\left(-s^2\pm\sqrt{\frac{p}{10s}+\frac{4s^4}{5}}\right)^{\frac{1}{2}}$

13. To find two numbers of which the product is given p, and also the product P, of the sums when each is increased by a given number (a and b).

$$\text{Ans. } x = \frac{P - p - ab}{2b} \pm \sqrt{\left(\left(\frac{P - p - ab}{2b}\right)^2 - \frac{ap}{b}\right)}.$$

14. To find two numbers such, that their sum, their product, and the difference of their squares, shall be all equal.

$$\text{Ans. } y = \frac{1 \pm \sqrt{5}}{2}, \text{ and } x = \frac{3 \pm \sqrt{5}}{2}.$$

15. Given the sum a, of two numbers, and the sum of their square roots b; to find the numbers.

$$\text{Ans. } x = \tfrac{1}{2}a \pm \tfrac{1}{2}b\sqrt{2a - b^2}.$$

16. Given the excess of the product of two numbers above their sum a, and also the sum of their squares b; to find the numbers.

$$\text{Ans. The greater} = \frac{s + \sqrt{(s^2 - 4p)}}{2}, \text{ and the less} = \frac{s - \sqrt{(s^2 - 4p)}}{2},$$

where s and p are their sum and product, and can easily be obtained from the question.

17. Given the sum s, of three numbers, of which the square of the greatest is equal to the squares of the other two, and also the continued product p, of the three numbers; to find the numbers.

Ans. The greatest is $\frac{s^2 \pm \sqrt{s^4 - 16sp}}{4s}$; the sum of the two less is $\frac{3s^2 \pm \sqrt{s^4 - 16sp}}{4s}$; and their product is $\frac{s^2 \pm \sqrt{s^4 - 16sp}}{4}$.

18. Let p be the given product of the two lesser numbers, the rest as before; to find the numbers.

Ans. The greatest is $\frac{s^2 - 2p}{2s}$, the sum of the two less is $\frac{s^2 + 2p}{2s}$, and their difference is $\frac{(s^4 - 12s^2p + 4p^2)^{\frac{1}{2}}}{2s}$.

19. Let, as before, the square of the greatest be equal to the squares of the other two, and the square of the middle one equal to the product of the greatest and least, and let the sum s of the three be given; to find each of them.

$$\text{Ans. The greatest} = \frac{s}{4}(\sqrt{5} + 1 - \sqrt{2\sqrt{5} - 2}).$$

20. Suppose still the square of the greatest equal to the

squares of the other two, and let the difference of the squares of the two least be equal to the product of the greatest by a given multiplier m, also the difference of the two least is given $=d$; to find the numbers.

Ans. The greatest is $=\frac{d^2}{\sqrt{2d^2-m^2}}$, or putting $n^2=2d^2-m^2$, it is $=\frac{d^2}{n}$, the next is $=\frac{(m+n)d}{2n}$, and the least is $=\frac{(m-n)d}{2n}$.

PROGRESSIONS.

A SERIES of quantities, which increase or decrease by a common difference, is called an Arithmetical Progression; as, 2, 5, 8, 11, &c., or 88, 85, 82, &c.

A series of quantities, which increase by a constant multiplier, or decrease by a common divisor, is called a Geometrical Progression; as, 2, 8, 32, 128, &c., or 567, 189, 63, &c.

The greatest and least terms are called the Extremes, and the other terms the Means.

ARITHMETICAL PROGRESSION.

If a represent the least term, y the greatest, d the common difference, and n the number of terms, any arithmetical progression may be expressed thus: $a, a+d, a+2d, a+3d$, &c. ascending; or $y, y-d, y-2d, y-3d$, &c. descending.

From these expressions it appears that the coefficient of d in any term is less by 1 than the number of that term.

PROP. I. The difference between the extremes is equal to the common difference, multiplied by the number of terms *minus* one. For the coefficient of d in the nth term is $n-1$.

Cor. Hence $y=a+(n-1)d$, and $a=y-(n-1)d$.

PROP. II. The sum of the extremes is equal to the sum of any two terms equally distant from them.

For any term exceeds the least, as much as its corresponding term is less than the greatest. Thus, if the series ascend from a to y, the whole will be $a, a+d, a+2d$, &c., $y-2d, y-d, y$; where the sum of any two corresponding terms is $a+y$.

Cor. The double of any term is equal to the sum of any two terms equally distant from it.

PROP. III. The sum of any number of terms is equal to the sum of the extremes multiplied by half the number of terms.

For by adding the extremes, and every two equally distant from them, we obtain equal sums, of which the number is half the number of terms of the series.

Cor. 1. Hence if $s =$ sum of the series, $s = (a+y)\frac{n}{2}$.

Cor. 2. If the number of terms be odd, and m the middle one, then $s = nm$; for $2m = a + y$.

Cor. 3. In a series of natural numbers, 1, 2, 3, &c. n, the sum $s = n \times \frac{n+1}{2}$; for n is the greatest term, and $n+1$ the sum of the extremes.

Cor. 4. In a series of even numbers, 2, 4, 6, &c., $s = n(n+1)$; for this series is $2 \times (1+2+3+$ &c.)

Cor. 5. In a series of odd numbers, beginning at 1, as 1, 3, 5, &c., $s = n^2$; for the sum of the extremes is double the number of terms.

1. Required the 12th term of the series 5, 8, 11, &c.

Here $n = 12, a = 5, d = 3$; therefore $y = 5 + 11 \times 3 = 38$.

2. Required the 7th term of the series 182, 178, 174, &c.

Here $n = 7, y = 182, d = 4$; therefore $a = 182 - 6 \times 4 = 158$.

3. Required the sum of 12 terms of the series 3, 8, 13, &c.

Here $a = 3, d = 5, n = 12, y = 3 + 11 \times 5 = 58$, and $s = (58 + 3)6 = 366$.

4. Required the sum of 14 terms of the series 89, 85, 81, &c.

Here $a = 89 - 13 \times 4 = 37$, and $s = (89 + 37)7 = 882$.

From these propositions any two of the five things mentioned may be found, if the other three be given. The theorems for finding them are expressed in the following Table:—

USE OF THE TABLE.

1. Let the least term be 7, the common difference 2, and the sum of the series 567. Required the greatest, and the number of terms.

$\sqrt{(567 \times 8 \times 2 + \overline{14 - 2}|^2)} = \sqrt{(9072 + 144)} = \sqrt{9216} = 96$, and $\frac{96-2}{2} = 47$, the greatest term; and $\frac{96 - 14 + 2}{2 \times 2} = 21$, the number of terms.

2. Given the least term 5, the number of terms 30, and the sum of the series 1455; to find the greatest term and the common difference.

$\frac{1455 \times 2}{30} - 5 = 92$ the greatest, $\frac{1455 - 5 \times 30}{15 \times 29} = 3$ the difference.

FORMULÆ FOR FINDING THE OTHER QUANTITIES.

Given.	Least $=a$.	Greatest $=y$.	Difference $=d$.	Number of Terms $=n$.	Sum $=s$.
a, y, n			$\frac{y-a}{n-1}$		$\frac{1}{2}n(y+a)$
a, d, n		$a+(n-1)d$			$\frac{1}{2}n(2a+(n-1)d)$
a, n, s		$\frac{2s}{n}-a$	$\frac{s-an}{\frac{1}{2}n(n-1)}$		
y, n, s	$\frac{2s}{n}-y$		$\frac{ny-s}{\frac{1}{2}n(n-1)}$		
y, n, d	$y-(n-1)d$				$\frac{1}{2}n(2y-(n-1)d)$
d, n, s	$\frac{s}{n}-\frac{n-1}{2}d$	$\frac{s}{n}+\frac{n-1}{2}d$			
a, y, s			$\frac{y^2-a^2}{2s-y-a}$	$\frac{2s}{y+a}$	
a, y, d				$\frac{y-a}{d}+1$	$\frac{(y+a)\times(y+d-a)}{2d}$
a, d, s		$\frac{(8ds+\overline{2a-d}\vert^2)^{\frac{1}{2}}-d}{2}$		$\frac{(8ds+\overline{2a-d}\vert^2)^{\frac{1}{2}}-2a+d}{2d}$	
y, d, s	$\frac{d\pm(\overline{2y+d}\vert^2-8ds)^{\frac{1}{2}}}{2}$			$\frac{2y+d\pm(\overline{2y+d}\vert^2-8ds)^{\frac{1}{2}}}{2d}$	

GEOMETRICAL PROGRESSION.

If a be the least term of a geometrical progression, y the greatest, r the common multiplier or divisor, called the common ratio, and n the number of terms, such a series, if ascending, may be expressed thus, a, ar, ar^2, ar^3, &c., or if descending, thus, y, $\frac{y}{r}$, $\frac{y}{r^2}$, $\frac{y}{r^3}$, &c.; where the exponent of r is one less than the number of the term.

Prop. I. The greatest term of a geometrical progression is equal to the least term, multiplied by that power of the common ratio, of which the exponent is the number of terms *minus* one.

For in the nth term, the exponent of r is $n-1$.

Therefore $y = ar^{n-1}$, and $a = \frac{y}{r^{n-1}}$.

Hence if $a = 1$, $y = r^{n-1}$.
Required the 8th term of the series 2, 6, 18, &c.
Here $a = 2$, $r = 3$, $n = 8$; therefore $2 \times 3^7 = 4374$.

Prop. II. The product of the extremes is equal to the product of any two terms equally distant from them.

For $a \times y = ar \times \frac{y}{r} = ar^2 \times \frac{y}{r^2}$, &c.

Cor. 1. The square of any term is equal to the product of any two terms equally distant from it.

Cor. 2. If there be four terms, the product of the means, divided by either extreme, gives the other; and if there be three terms, the square of the mean, divided by either extreme, gives the other.

1. Required a third proportional to 85 and 425. Ans. 2125.
2. . . . . . . a fourth proportional to 18, 54, 162. . 486.

Prop. III. If the sum of a geometrical progression be multiplied by the common ratio, and the series be subtracted from the product, the remainder will be equal to the excess of the product of the highest term by the ratio, above the least term.

For the whole series, except the least term, will be included in the product. Thus, if $a + ar + ar^2$, &c. $+ \frac{y}{r^2} + \frac{y}{r} + y = s$ be multiplied by r, it becomes $ar + ar^2$, &c. $+ \frac{y}{r} + y + yr = sr$; and subtracting the original series, we obtain $yr - a = sr - s$.

Whence $s = \frac{yr - a}{r - 1} = \frac{a(r^n - 1)}{r - 1}$.*

Cor. 1. The difference between any two adjacent terms is equal to the less multiplied by the ratio, wanting one.

Thus, $ar^3 - ar^2 = ar^2 \times (r - 1)$. Wherefore, if the difference of the extremes be multiplied by the greatest term but one, and divided by the difference between the two greatest terms, the quotient will be the sum of all the terms except the greatest. For the divisor is the product of the multiplier by $r - 1$.

Cor. 2. If the common ratio be 2, the difference of the extremes is the sum of all the terms except the greatest.

Cor. 3. If a descending series be interminate, the least term may be considered $= 0$, and the sum $= \frac{yr}{r - 1}$.

1. Required the 8th term of the series 4, 8, 16, &c.

$$4 \times 2^7 = 4 \times 128 = 512.$$

2. Required the sum of 12 terms of the series 1, 3, 9, 27, &c. $\frac{3^{12} - 1}{3 - 1} = \frac{531441 - 1}{2} = 265720.$

3. Required the sum of 8 terms of the series 1, $\frac{1}{3}$, $\frac{1}{9}$, &c. $\frac{1 - \overline{\frac{1}{3}}|^8}{1 - \frac{1}{3}} = \left(1 - \frac{1}{6561}\right) \times \frac{3}{2} = \frac{6560}{6561} \times \frac{3}{2} = \frac{3280}{2187}.$

4. Given the extremes a and y, and the sum of the series s, to find the common ratio and the number of terms.

Ans. $r = \frac{s - a}{s - y}$. Having found r, $r^{n-1} = \frac{y}{a}$. And in logarithms, where R, Y, and A represent the logarithms of r, y, and a, $(n - 1)\mathrm{R} = \mathrm{Y} - \mathrm{A}$, and $n = \frac{\mathrm{Y} - \mathrm{A} + \mathrm{R}}{\mathrm{R}}$.

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* In this formula r may represent any quantity, integral or fractional, except unity. If $r = 1$, there could be no progression; for every power of 1 is 1, and therefore the formula would be $\frac{a(1-1)}{1 - 1} = \frac{a \times 0}{0}$, a very improper expression. When a is multiplied by a quantity less than 1, the product is less than the multiplicand; and the less that the multiplier is taken, the less will the product be; so that $a \times 0 = 0$, or less than any quantity. Again, when a is divided by a quantity less than 1, the quotient is greater than a; and the less that the divisor is taken, the greater will the quotient be: therefore $\frac{a}{0}$ will be infinitely great, or greater than any quantity. To avoid this absurdity, divide first by the denominator, and then affix values to the quantities. If $ar^n - a$ be divided by $r - 1$, the quotient is $ar^{n-1} + ar^{n-2} + ar^{n-3} +$ &c; and if

QUESTIONS ON PROGRESSIONS.

1. To find four numbers in arithmetical progression, such, that their sum shall be 56, and the sum of their squares 864. Let the numbers be $x, x+y, x+2y, x+3y$, then their sum $4x+6y=56$, or $2x+3y=28$, and the sum of their squares $4x^2+12xy+14y^2=864$, from which subtract $\overline{2x+3y}|^2=28^2$, or $4x^2+12xy+9y^2=784$; the remainder gives $5y^2=80$, or $y=4$, and $x=8$; and the numbers are 8, 12, 16, 20.

2. To find three numbers in arithmetical progression, such, that their sum shall be 9, and the sum of their cubes 153. Let the numbers be $x-y, x, x+y$, then their sum $3x=9$, and the sum of their cubes $3x^3+6xy^2=153$.

Ans. The numbers are, 1, 3, 5.

3. To find three numbers in arithmetical progression, such, that their sum shall be 15, and the sum of the squares of the extremes 58. Ans. 3, 5, 7.

4. To find four numbers in arithmetical progression, such, that the sum of the extremes shall be 8, and the product of the means 15. Ans. 1, 3, 5, 7.

5. To find four numbers in arithmetical progression, such, that the sum of the squares of the means shall be 52, and the sum of the squares of the extremes 68. Ans. 2, 4, 6, 8.

6. A traveller goes 9 miles a-day: after 7 days another sets out after him, and travels 4 miles the first day, 5 miles the second, 6 miles the third, and so on. In what time will he overtake the first?

Here $\frac{8+x-1}{2}x=(x+7)9$. Ans. 18 days.

7. To find three numbers in geometrical progression, such, that their sum shall be 7, and the sum of their squares 21. Let x, y, z, be the numbers.

Then $xz=y^2$, $x+y+z=7$, $x^2+y^2+z^2=21$.

Ans. 1, 2, 4.

8. To find four numbers in geometrical progression, such, that their sum shall be 30, and that the greatest shall be equal to the sum of the means multiplied by $1\frac{1}{3}$.

Let x, xy, xy^2, xy^3, be the numbers. Ans. 2, 4, 8, 16.

$r=1$, it will be $a(1+1+1+1+\&c.)=na$, which, though not a geometrical progression, is a determined quantity. In like manner $\frac{x^2-a^2}{x-a}$ would be $\frac{0}{0}$, if x were $=a$; but if we divide first, the quotient will be $x+a$, which is $=2a$, when $x=a$. And many other cases may occur like these.

9. To find three numbers in geometrical progression, such, that their product shall be 64, and the sum of their cubes 584. Let x, xy, xy^2, be the numbers.

Then $x^3y^3 = 64$, $x^3 \times (1+y^3+y^6) = 584$. Ans. 2, 4, 8.

10. To find three numbers in geometrical progression, such, that the sum of the first and third shall be 52, and their product 100. Ans. 2, 10, 50.

11. To find two mean proportionals between 4 and 256.
Ans. 16 and 64

12. Given the sum of the squares a, of three numbers in arithmetical progression, and the excess of the square of the mean above the product of the extremes b; to find the numbers.

Ans. Comm. diff. $\sqrt{b}$, mean $\sqrt{\left(\frac{a-2b}{3}\right)}$.

13. Given the product of the extremes a, and the product of the means b, of four numbers in arithmetical progression; to find the numbers.

Ans. Com. diff. $\sqrt{\left(\frac{b-a}{2}\right)}$, least $\frac{1}{2}\left\{\sqrt{\left(\frac{9b-a}{2}\right)}-3\sqrt{\left(\frac{b-a}{2}\right)}\right\}$.

14. Given the number of terms n, of an arithmetical progression, their sum a, and the sum of their squares b; to find the terms. Let the terms be $x+y$, $x+2y$, $x+3y \ldots x+ny$.

Then $y = \left(\frac{12nb-12a^2}{n^2(n^2-1)}\right)^{\frac{1}{2}}$, and $x = \frac{a}{n} - \frac{n+1}{n}\left(\frac{3nb-3a^2}{n^2-1}\right)^{\frac{1}{2}}$.

15. Suppose two travellers set out at the same time from two places of which the distance is given, p. The miles travelled by the first per day form a decreasing arithmetical progression, of which the first term is given, a, and the common difference d. Those travelled by the second form an increasing series, of which the first term is b, and the common difference c. In what time will they meet?

Let $a+b=m$, and $c-d=n$.

Ans. $\frac{1}{2}-\frac{m}{n} \pm \sqrt{\left(\frac{2p}{n}+\left(\frac{m}{n}-\frac{1}{2}\right)^2\right)}$, or $\frac{p}{m}$, (if $n=0$).

16. Given the sum s of five numbers in geometrical progression, and the sum of their squares a; to find the numbers.

Suppose $v =$ sum of the first and third, then $v = \frac{s}{2} - \frac{a}{2s}$, and

the second $= \sqrt{\left(v^2+\left(\frac{s-v}{2}\right)^2\right)} - \frac{s-v}{2}$.

INTEREST AND ANNUITIES.

In **Simple Interest**, the interest is computed on the principal only. Let $p=$ principal or money lent, $t=$ time, $r=$ rate or interest of £1 for the time one, $i=$ interest for the whole time, $a=$ amount or sum of principal and interest; then $rt=$ interest of £1 for the time t, and $1+rt$ the amount of £1, and $p\times(1+rt)=p+prt=p+i=a$ the amount of the whole; from which equations the value of any of the quantities concerned may be found in terms of the others.

In **Compound Interest**, the interest at each term of payment is added to the principal, and the amount is the principal for the next term. Let $R=1+r$ the amount of £1 for the first term, it will be the principal for the next term, and the interest upon it will be Rr, and the amount $Rr+R=R(r+1)=R^2$ will be the principal for the next term. In like manner we find that the amounts at the end of the following terms will be R^3, R^4, &c.; and at the end of the time t it will be R^t, and for the principal p it will be pR^t the amount, and the interest will be $pR^t-p=i=a-p$; from which equations any of the quantities may be expressed in terms of the rest.

Of Annuities. If $m=$ principal, which yields £1 of annual interest at the given rate, then $mR^t-m=$ interest of this principal for the time t, which will therefore be the amount of an annuity of £1 for that time. But $m=\frac{1}{r}$, and therefore the amount will be $\frac{R^t-1}{r}$; and for any annuity n, it will be $\frac{nR^t-n}{r}=a$. And if p be equal to the present value of this annuity, then $\frac{nR^t-n}{r}=pR^t$, and $p=\frac{n-\frac{n}{R^t}}{r}$, where $\frac{1}{R^t}$ is the present worth of £1.

Of Reversions. When the annuity does not commence till some time after this, it is said to be in reversion. The amount, if it were to commence just now, would be $n\times\frac{R^t-1}{r}$; but if it commence s years after this, it will be $\frac{n}{R^s}\times\frac{R^t-1}{r}=a$, and the present worth $p=\frac{n}{R^s}\times\frac{1-\frac{1}{R^t}}{r}$.

From these equations any of the quantities may be expressed in terms of the others.

IN A FREEHOLD ESTATE, the value $y=\frac{1}{r}$ when the rent is £1, and it commences just now; and $\frac{1}{R^s r}$ is its value, when it does not commence till s years after this, y is called the year's purchase or perpetuity, and $ay=v$ the value of the estate, of which the rent is a, and $\frac{ay}{R^s}$ is the value in reversion.

ANNUITIES ON LIVES. Adopting Mr De Moivre's hypothesis, that of a certain number born at one time, one dies every year until the whole is extinct, a supposition which agrees nearly with observation, for ages between 10 and 60. An annuity of £1 for a given life will be the sum of the series $\frac{n-1}{nr}+\frac{n-2}{nr^2}+\frac{n-3}{nr^3}+$ &c., continued to $\frac{n-n}{nr^n}$, where n is the complement of the age, or what it wants of the age at which the oldest dies, which he supposed to be 86, and r the amount of £1 for a year. This sum is $\frac{\left(n-1+\frac{1}{r^n}\right)r-n}{n(r-1)^2}=\frac{n-1-q}{n(r-1)}$, supposing q to be the present worth of an annuity of £1 for $n-1$ years.

Again, the value of an annuity for two joint lives, of which the complements are n and m (the greatest m), will be $\frac{n-1}{n}\times\frac{m-1}{mr}+\frac{n-2}{n}\times\frac{m-2}{mr^2}+\frac{n-3}{n}\times\frac{m-3}{mr^3}+$ &c. continued to $\frac{n-n}{n}\times\frac{m-n}{mr^n}$, of which the sum is $\frac{1}{r-1}+$ $\frac{(m-n)\frac{1}{r^n}-(m+n)}{mn}\times\frac{r}{(r-1)^2}+\frac{\left(1-\frac{1}{r^n}\right)(r+1)r}{mn\times(r-1)^3}$; or if $s=$ value of the oldest life, the value of the two lives is $\frac{(n-1)p-s\times(2p+1-(m-n))}{m}$, where $p=$ perpetuity.

If a question occur which involves both interest and annuities, an equation may be found answering to it by comparing with one another the values of the quantities found separately.

1. What will £1000 amount to in 10 years, at 5 per cent. compound interest? Ans. £1628, 17s. 9½d.

2. What principal will, in 15 years, amount to £2000, at 4 per cent. compound interest? Ans. £1110, 10s. 7½d.

3. In what time will £200 amount to £318, 16s., at 6 per cent. compound interest? Ans. 8·0016 years.

4. In what time will a sum of money double itself, at 4 per cent. compound interest? $(1{\cdot}04)^t = 2$.
Ans. 17·673 years.

5. Required the amount of £20 annuity for 41 years, at 5 per cent.? Ans. £2556, 15s. 11d.

6. What annuity will, in 7 years, amount to £79, at 4 per cent.? Ans. £10·0022.

7. What is the value of an annuity of £20, for a life of 54 years of age, at 4 per cent.? Ans. £209·55469.

8. What is the value of an annuity of £20, during the joint lives of two persons, whose ages are 35 and 25 years, at 4 per cent.? Ans. £221·9176.

9. When 12 years of a lease of 21 years were expired, a renewal for the same term was granted for £1000. Eight years of that lease are now expired, and it is required what sum should be paid for a corresponding renewal of the lease, reckoning 5 per cent. compound interest.

From the first transaction, find the annuity $n =$ £175·029955, and from it find p, the present worth of the annuity in reversion = £599·9294.

OF SERIES.

A Series is an assemblage of terms, which continually increase or decrease according to a certain law, as the arithmetical and geometrical series treated of before.

A Converging Series is that of which the terms continually decrease, and a Diverging Series is one of which the terms continually increase.

Series are obtained by division, by the extraction of roots, and by various other operations.

1. Thus, $\frac{ax}{a-x} = x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3} +$ &c., where the exponents increase by one.

2. Also $\sqrt{a^2+x^2} = a + \frac{x^2}{2a} - \frac{x^4}{2.4a^3} + \frac{3x^6}{2.4.6a^5} - \frac{3.5x^8}{2.4.6.8a^7} + \frac{3.5.7x^{10}}{2.4.6.8.10a^9} +$ &c.

OF THE BINOMIAL THEOREM.

The Binomial Theorem is a general formula, discovered by Sir Isaac Newton, whereby any power or root of a binomial

may be obtained without performing the involution or extraction. The power or root found by this theorem is called the development or expansion of the binomial.

The following is the form in which it was first proposed by Newton:—

$$(P+PQ)^{\frac{m}{n}} = P^{\frac{m}{n}}\left\{1+\frac{m}{n}Q+\frac{m}{n}.\frac{m-n}{2n}Q^2+\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}Q^3 + \frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}.\frac{m-3n}{4n}Q^4+\text{\&c.}\right\}$$

Or

$$(P+PQ)^{\frac{m}{n}}=P^{\frac{m}{n}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ+\frac{m-2n}{3n}CQ+\frac{m-3n}{4n}DQ+\text{\&c.}$$

Where P is the first term of the binomial, Q the second term divided by the first, $\frac{m}{n}$ the exponent of the power or root, and A, B, C, D, &c. the terms immediately preceding those in which they are first found, including their signs $+$ or $-$.

This theorem may be applied to any particular case, by substituting the quantities in the given example for P, Q, m, and n, in the formula, and then finding the result.

NOTE. When the exponent of the binomial is a whole number, the series will terminate, but when it is a negative or fractional number, the series will not terminate, but proceed on, and become more convergent the smaller the second term is with respect to the first.

Required the development of $\frac{x^2}{(x^2-y)^{\frac{1}{2}}}$ in a series.

Here $\frac{x^2}{(x^2-y)^{\frac{1}{2}}} = x^2(x^2-y)^{-\frac{1}{2}}$, $P = x^2$, $Q = -\frac{y}{x^2}$, $m = -1$, and $n = 2$; hence

$$P^{\frac{m}{n}} = (x^2)^{\frac{m}{n}} = (x^2)^{-\frac{1}{2}} \qquad = \frac{1}{x} \qquad = A.$$

$$\frac{m}{n}AQ = -\frac{1}{2}\times\frac{1}{x}\times-\frac{y}{x^2} \qquad = \frac{y}{2x^3} \qquad = B.$$

$$\frac{m-n}{2n}BQ = \frac{-1-2}{4}\times\frac{y}{2x^3}\times-\frac{y}{x^2} \qquad = \frac{3y^2}{2.4x^5} \qquad = C.$$

$$\frac{m-2n}{3n}CQ = \frac{-1-4}{6}\times\frac{3y^2}{2.4x^5}\times-\frac{y}{x^2} = \frac{3.5y^3}{2.4.6x^7} \qquad = D.$$

$$\frac{m-3n}{4n}DQ = \frac{-1-6}{8}\times\frac{3.5y^3}{2.4.6x^7}\times-\frac{y}{x^2} = \frac{3.5.7y^4}{2.4.6.8x^9} = E.$$

&c. &c. &c.

$$\therefore \frac{1}{(x^2-y)^{\frac{1}{2}}} = \frac{1}{x} + \frac{y}{2x^3} + \frac{3y^2}{2.4x^5} + \frac{3.5y^3}{2.4.6x^7} + \frac{3.5.7y^4}{2.4.6.8x^9} + \&c.$$

$$\text{and } \frac{x^2}{(x^2-y)^{\frac{1}{2}}} = x + \frac{y}{2x} + \frac{3y^2}{2.4x^3} + \frac{3.5y^3}{2.4.6x^5} + \frac{3.5.7y^4}{2.4.6.8x^7} + \&c.$$

Required the value of $9^{\frac{1}{3}}$ in an infinite series.

Here $9^{\frac{1}{3}} = (8+1)^{\frac{1}{3}}$ $\therefore$ $P=8$, $Q=\frac{1}{8}$, $m=1$, and $n=3$; whence

$$P^{\frac{m}{n}} = 8^{\frac{m}{n}} = 8^{\frac{1}{3}} \qquad = 2 \qquad = A.$$

$$\frac{m}{n}AQ = \frac{1}{3} \times 2 \times \frac{1}{2^3} \qquad = \frac{1}{3.2^2} \qquad = B.$$

$$\frac{m-n}{2n}BQ = \frac{1-3}{6} \times \frac{1}{3.2^2} \times \frac{1}{2^3} \qquad = -\frac{1}{3.6.2^4} \qquad = C.$$

$$\frac{m-2n}{3n}CQ = \frac{1-6}{9} \times -\frac{1}{3.6.2^4} \times \frac{1}{2^3} = \frac{5}{3.6.9.2^7} \qquad = D.$$

$$\frac{m-3n}{4n}DQ = \frac{1-9}{12} \times \frac{5}{3.6.9.2^7} \times \frac{1}{2^3} = -\frac{5.8}{3.6.9.12.2^{10}} = E.$$

&c. &c. &c.

$$\therefore 9^{\frac{1}{3}} = 2 + \frac{1}{3.2^2} - \frac{1}{3.6.2^4} + \frac{5}{3.6.9.2^7} - \frac{5.8}{3.6.9.12.2^{10}} + \&c.$$

1. Expand $(y^2 - x^2)^{\frac{3}{4}}$ into an infinite series.

$$\text{Ans. } \frac{1}{\sqrt{y}}\left(y^2 - \frac{3x^2}{2^2} - \frac{3x^4}{2^5y^2} - \frac{5x^6}{2^7y^4} - \frac{5.9x^8}{2^{11}y^6} - \&c.\right)$$

2. Expand $\left(\frac{x^3}{x^3+y^3}\right)^{\frac{1}{3}}$ into an infinite series.

$$\text{Ans. } 1 - \frac{y^3}{3x^3} + \frac{2y^6}{3^2x^6} - \frac{2.7y^9}{3^4x^9} + \&c.$$

3. Develop $\left(\frac{x-y}{x+y}\right)^{\frac{1}{2}}$ in an infinite series.

$$\text{Ans. } 1 - \frac{y}{x} + \frac{y^2}{2x^2} - \frac{y^3}{2x^3} + \frac{3y^4}{2.4x^4} - \frac{3y^5}{2.4x^5} + \frac{3.5y^6}{2.4.6x^6} - \&c.$$

4. Develop $\frac{x}{(x \mp y)^{\frac{1}{3}}}$ in an infinite series.

$$\text{Ans. } x^{\frac{2}{3}}\left(1 \pm \frac{y}{3x} + \frac{4y^2}{3.6x^2} \pm \frac{4.7y^3}{3.6.9x^3} + \frac{4.7.10y^4}{3.6.9.12x^4} \pm \&c.\right)$$

5. Required the value of $\sqrt[3]{7}$ in an infinite series.

$$\text{Ans. } 2 - \frac{1}{3.2^2} - \frac{1}{3.6.2^4} - \frac{5}{3.6.9.2^7} - \frac{5.8}{3.6.9.12.2^{10}} - \&c.$$

6. Expand $(1-a)^{\frac{2}{5}}$ into an infinite series.

$$\text{Ans. } 1-\frac{2a}{5}-\frac{2.3a^2}{5.10}-\frac{2.3.8a^3}{5.10.15}-\frac{2.3.8.13a^4}{5.10.15.20}-\&c.$$

7. Required the development of $(b^2+x)^{\frac{1}{2}}$ in a series.

$$\text{Ans. } b+\frac{x}{2b}-\frac{x^2}{2.4b^3}+\frac{3x^3}{2.4.6b^5}-\frac{3.5x^4}{2.4.6.8b^7}+\frac{3.5.7x^5}{2.4.6.8.10b^9}-\&c.$$

OF THE METHOD OF INDETERMINATE COEFFICIENTS.

This is a general method of obtaining series from fractional and other expressions without either performing the division or extracting the root.

Assume a series with unknown, but constant, coefficients, and having the exponents of x increasing or decreasing in the same way as if the operation was performed at length; then make this series equal to the given expression, and, clearing the equation of fractions, bring all the terms to one side, so as to make the equation $=0$; next make the first term and the coefficients of the several powers of x each $=0$,* and there will arise as many independent equations as there are unknown coefficients, from which their values may be found and substituted for them in the assumed series.

Let it be required to expand $\frac{a}{b+x}$ into a series.

Assume $\frac{a}{b+x}=\text{A}+\text{B}x+\text{C}x^2+\text{D}x^3+$ &c.; then multiplying both sides by $b+x$, and transposing a, we obtain $\text{A}b-a+(\text{B}b+\text{A})x+(\text{C}b+\text{B})x^2+(\text{D}b+\text{C})x^3+$ &c. $=0$, an equation which must be true whatever be the value of x.

Now, making the first term and the coefficients of the several powers of x each $=0$, we have $\text{A}b-a=0$, or $\text{A}=\frac{a}{b}$; $\text{B}b+\text{A}=0$, or $\text{B}=\frac{\text{A}}{b}=-\frac{a}{b^2}$; $\text{C}b+\text{B}=0$, or $\text{C}=\frac{\text{B}}{b}=+\frac{a}{b^3}$; $\text{D}b+c=0$, or $\text{D}=\frac{\text{C}}{b}=-\frac{a}{b^4}$, &c. And substituting these values of A, B, C, D, &c. in the assumed series,

* If the series $(a+b)x+(c+d)x^2+(e+f)x^3$, &c. continued indefinitely, be always $=$ nothing, whatever be the value of x, then the coefficient of any one power of x is $=0$; that is, $a+b=0$, $c+d=0$, &c. For if the equation be divided by x, then $a+b+(c+d)x+(e+f)x^2=0$. Here let $x=0$, then $a+b=0$; therefore $(c+d)x+(e+f)x^2=0$, whatever be the value of x; and proceeding in the same way we find $c+d=0$, and so on.

we get $\frac{a}{b+x}=\frac{a}{b}-\frac{ax}{b^2}+\frac{ax^2}{b^3}-\frac{ax^3}{b^4}+$ &c., in which it is obvious that the signs are alternately $+$ and $-$, and the exponents, both in the numerator and denominator, increase continually by 1, that of x in the numerator being always 1 less than that of b in the denominator.

2. Expand $\frac{a^2}{a^2+2ax-x^2}$ into a series.

Ans. $1-\frac{2x}{a}+\frac{5x^2}{a^2}-\frac{12x^3}{a^3}+$ &c.

3. Expand $\sqrt{(a^2-x^2)}$ into a series.

Ans. $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-$ &c.

4. Expand $\frac{1+2x}{1-x-x^2}$ into a series.

Ans. $1+3x+4x^2+7x^3+11x^4+18x^5+$ &c.

This is a recurring series, in which each of the coefficients after the second is the sum of the two preceding ones.

5. Expand $\sqrt{(1-a)}$ into a series.

Ans. $1-\frac{a}{2}-\frac{a^2}{2.4}-\frac{3a^3}{2.4.6}-\frac{3.5a^4}{2.4.6.8}-\frac{3.5.7a^5}{2.4.6.8.10}-$ &c.

OF THE SUMMATION AND INTERPOLATION OF SERIES.

The summation of series is the method of finding a terminated expression equal to the whole series, and interpolation is the method of finding any term of an infinite series without producing all the rest.

OF THE DIFFERENTIAL METHOD.

The differential method consists in finding from the successive differences of the terms of a series any intermediate term or the sum of the whole series.

PROB. I. To find the several orders of differences.

Let $a+b+c+d+e+$ &c. be any series; subtract each term from the one following it, and the differences $-a+b$, $-b+c$, $-c+d$, $-d+e$, &c. will form a new series, called the first order of differences. Again, subtract each term of this new series from the one that follows it, and the differences $a-2b+c$, $b-2c+d$, $c-2d+e$, &c. will form another series, called the second order of differences. Proceed in like manner for the third, fourth, fifth, &c. order of differences, until they at last become equal to 0, or are carried as far as is required.

NOTE. When the several terms of the series continually increase, the differences will be all positive; but when they decrease, the differences will be alternately negative and positive.

Required the several orders of differences of the series 1, 6, 20, 50, 105, 196, &c.

1,	6,	20,	50,	105,	196,	&c. the given series.	
	5,	14,	30,	55,	91,	&c. 1st differences.	
		9,	16,	25,	36,	&c. 2d	do.
			7,	9,	11,	&c. 3d	do.
				2,	2,	&c. 4th	do.
					0,	&c. 5th	do.

2. Required the several orders of differences of the series $1, 2^2, 3^2, 4^2, 5^2$, &c. Ans. 1st diff. 3, 5, 7, 9, 11, &c.; 2d diff. 2, 2, 2, 2, &c.; 3d diff. 0.

3. Required the several orders of differences of the series of cubes $1^3, 2^3, 3^3, 4^3, 5^3$, &c. Ans. 1st diff. 7, 19, 37, 61, &c.; 2d diff. 12, 18, 24, &c.; 3d diff. 6, 6, &c.; 4th diff. 0.

PROB. II. To find the first term of any order of differences.

Let $d', d'', d''', d^{\text{IV}}$, &c. represent the first terms of the 1st, 2d, 3d, 4th, &c. orders of differences; then $d' = -a+b$, $d'' = a-2b+c$, $d''' = -a+3b-3c+d$, $d^{\text{IV}} = a-4b+6c-4d+e$, &c. from which it is obvious that the coefficients of the several terms of any order of differences are respectively the same as those of the terms of an expanded binomial, and are obtained in the same manner; for the terms which are subtracted are actually added, but with contrary signs. Hence we infer that d^n, or the first difference of the nth order of differences, is

$$\pm a \mp nb \pm n.\frac{n-1}{2}c \mp n.\frac{n-1}{2}.\frac{n-2}{3}d \pm \text{\&c. to } n+1$$

terms, in which formula the upper signs must be taken when n is an even number, and the under when n is an odd number.

1. Required the first of the fifth order of differences of the series 6, 9, 17, 35, 63, 99, 148, &c.

Here a, b, c, d, e, f, &c. $= 6, 9, 17, 35, 63, 99$, &c. and $n=5$

$$\therefore -a+nb-\frac{n(n-1)}{2}c+\frac{n(n-1)(n-2)}{2.3}d-\frac{n(n-1)(n-2)(n-3)}{2.3.4}e$$

$$+\frac{n(n-1)(n-2)(n-3)(n-4)}{2.3.4.5}f = -a+5b-\frac{5.4}{2}c+\frac{5.4.3}{2.3}d$$

$-\frac{5.4.3.2}{2.3.4}e+\frac{5.4.3.2.1}{2.3.4.5}f=-6+45-170+350-315+99$
$=494-491=+3.$

2. Required the first of the sixth order of differences of the series 3, 6, 11, 17, 24, 36, 50, 72, &c. Ans. -14.

3. Required the first of the eighth order of differences of the series 1, 3, 9, 27, 81, &c. Ans. 256.

4. Required the first of the fifth order of differences of the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, &c. Ans. $-\frac{1}{32}$.

PROB. III. To find the nth term of the series a, b, c, d, e, f, &c.

Since $d'=-a+b$, therefore $b=a+d'$, and, in the same manner, we find $c=a+2d'+d''$, $d=a+3d'+3d''+d'''$, $e=a+4d'+6d''+4d'''+d^{\text{IV}}$, &c.; whence the nth term is $=a+\frac{n-1}{1}d'+\frac{n-1}{1}.\frac{n-2}{2}d''+\frac{n-1}{1}.\frac{n-2}{2}.\frac{n-3}{3}d'''+$&c.

1. Required the 7th term of the series 3, 5, 8, 12, 17, &c.

Here $d'=2$, $d''=1$, $d'''=0$, and $n=7$ $\therefore$ $a+\frac{n-1}{1}d'$ $+\frac{n-1}{1}.\frac{n-2}{2}d''=3+\frac{7-1}{1}.2+\frac{7-1}{1}.\frac{7-2}{2}.1=3+12+15$ $=30=$ the 7th term.

2. Required the 9th term of the series 1, 5, 15, 35, 70, &c. Ans. 495.

3. Required the 10th term of the series 1, 3, 6, 10, 15, 21, &c. Ans. 55.

PROB. IV. To find the sum of n terms of the series a, b, c, d, e, &c.

If we add the values of a, b, c, &c. as found in the last problem, we obtain $2a+d'=a+b$, $3a+3d'+d''=a+b+c$, $4a+6d'+4d''+d'''=a+b+c+d$, &c.; whence it is manifest that the sum of n terms must be $na+n.\frac{n-1}{2}d'$ $+n.\frac{n-1}{2}.\frac{n-2}{3}d''+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}d'''+$ &c.

NOTE. When the differences become at last $=0$, any term, or the sum of any number of terms, can be accurately found; but when the differences do not vanish, the formulæ in this and the preceding problem give only an approximation, which will come nearer the truth as the differences diminish.

1. Required the sum of 8 terms of the series 2, 5, 10, 17, &c. Here $n=8$, $a=2$, $d'=3$, $d''=2$, and $d'''=0$; hence

$na+n.\frac{n-1}{2}d'+n.\frac{n-1}{2}.\frac{n-2}{3}d'' = 8.2+8.\frac{7}{2}.3+8.\frac{7}{2}.\frac{6}{3}.2$ $=16+84+112=212=$ the sum of 8 terms.

2. Required the sum of 12 terms of the series 21, 56, 126, 252, 462, 792, &c. Ans. 27125.

3. Required an expression for the sum of n terms of the fourth order of figurate numbers, 1, 4, 10, 20, 35, &c.

Here $d'=3$, $d''=3$, $d'''=1$, and $d^{\mathrm{IV}}=0$; hence $s=n+n.\frac{n-1}{2}.3+n.\frac{n-1}{2}.\frac{n-2}{3}.3+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.1$, which, reduced, gives $s=n.\frac{n+1}{2}.\frac{n+2}{3}.\frac{n+3}{4}$; where it may be observed that the number of factors in the formula, and the order of differences which become $=0$, are the same with the order of the figurates.

4. Required the sum of 12 terms of the fourth order of figurates 1, 4, 10, 20, 35, &c. Ans. 1365.

5. Required an expression for the sum of n terms of the series of squares $(m\pm a)^2+(m\pm 2a)^2+(m\pm 3a)^2$, &c. $+(m\pm na)^2$.

Ans. $nm^2\pm n.\frac{n+1}{2}.2ma+n.\frac{n+1}{2}.\frac{2n+1}{3}.a^2$.

6. Required the sum of 12 terms of the series $3^2+5^2+7^2+9^2+$ &c. Ans. 2924.

7. Required an expression for the sum of n terms of the series of cubes $(m\pm a)^3+(m\pm 2a)^3+(m\pm 3a)^3$, &c. $+(m\pm na)^3$.

Ans. $nm^3\pm n.\frac{n+1}{2}.3m^2a+n.\frac{n+1}{1}.\frac{2n+1}{2}.ma^2\pm n^2\left(\frac{n+1}{2}\right)^2a^3$.

8. Required the sum of 9 terms of the series $3^3+6^3+9^3+12^3+$ &c. Ans. 54675.

9. Required an expression for the sum of n terms of the series of products $pq+(p-1)\times(q-1)+(p-2)\times(q-2)+(p-3)\times(q-3)+$ &c.

Ans. $\frac{3pq^2+3pq-q^3+q}{6}$, when $n=q+1$, and the series is complete; but if the number of terms n, be less than q, the expression will be $npq-n.\frac{n-1}{2}(p+q)+n.\frac{n-1}{2}.\frac{2n-1}{3}$.

10. Required the sum of 6 terms of the series $9\times8+8\times7+7\times6+6\times5$, &c. Ans. 232.

PROB. V. To find by interpolation any intermediate term

of the series *a, b, c, d, e,* &c. whose terms are equidistant from each other.

Let x be the place in the series, of any term y that is to be interpolated, and the first terms of the several orders of differences as before; then will

$$y = a + xd' + x.\frac{x-1}{2}.d'' + x.\frac{x-1}{2}.\frac{x-2}{3}.d''' + \&c.$$

NOTE. In finding the differences, each term is taken from the one which follows it, so that, when the former is the greater, the difference is *negative;* hence, in applying the formula to practice, the *signs* of the differences must be carefully attended to.

1. Required the logarithmic sine of 1° 1′ 40″, having given the log. sines of 1° 0′, 1° 1′, 1° 2′, and 1° 3′.

Series.	Log. Sines.	1st Diff.	2d Diff.	3d Diff.
1° 0′	8·241855			
		7178		
1 1	8·249033		—117	
		7061		+4
1 2	8·256094		—113	
		6948		
1 3	8·263042			

Here $a = 8{\cdot}241855, x = (1°\,1'\,40'' - 1°\,0') = 1'\,40'' = 1\frac{2}{3} = \frac{5}{3}, d' = 7178, d'' = -117,$ and $a''' = +4$; whence $y = a + xd' + x.\frac{x-1}{2}d'' + x.\frac{x-1}{2}.\frac{x-2}{3}.d''' = a + \frac{5}{3}d' + \frac{5}{3}.\frac{2}{6}.d'' - \frac{5}{3}.\frac{2}{6}.\frac{1}{9}d''' = 8{\cdot}241855 + 011963 - 000065 - 000000 = 8{\cdot}253753 =$ log. sine of 1° 1′ 40″.

2. Given the log. sines of 2° 4′, 2° 5′, 2° 6′, and 2° 7′, to find the log. sine of 2° 6′ 30″. Ans. 8·565719.

3. Given the series $\frac{1}{40}, \frac{1}{41}, \frac{1}{42}, \frac{1}{43}, \frac{1}{44}$, &c. to find the term which falls in the middle between $\frac{1}{42}$ and $\frac{1}{43}$. Ans. $\frac{2}{85}$.

4. Given the natural signs of 88° 54′, 88° 55′, 88° 56′, 88° 57′, 88° 58′, and 88° 59′, to find the natural sine of 88° 57′ 50″. Ans. ·999835.

PROB. VI. To find any intermediate term of the series *a, b, c, d, e,* &c. by interpolation, when the first differences of any order are small, or become = 0.

Find the value of the unknown quantity in the equation which stands opposite the given number of terms in the following table, and it will be the term required.

$$
\begin{array}{r|l}
1. & a - b = 0 \\
2. & a - 2b + c = 0 \\
3. & a - 3b + 3c - d = 0 \\
4. & a - 4b + 6c - 4d + e = 0 \\
5. & a - 5b + 10c - 10d + 5e - f = 0. \\
 & \&c. \qquad \&c. \qquad \&c. \\
n. & a - nb + n.\frac{n-1}{2}.c - n.\frac{n-1}{2}.\frac{n-2}{3}.d +
\end{array}
$$

$$n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.e - \&c. = 0.^*$$

1. Given the logarithms of 201, 202, 203, and 205, to find that of 204.

Here the given number of terms is 4, and opposite 4 in the table stands $a - 4b + 6c - 4d + e = 0$, or $d = \frac{a + 6c + e - 4b}{4}$.

Now, using the logarithms of the given terms, we have

$$
\left.\begin{array}{l}
\text{Log. } a = 2{\cdot}303169 \\
\quad\; b = 2{\cdot}305351 \\
\quad\; c = 2{\cdot}307496 \\
\quad\; e = 2{\cdot}311754
\end{array}\right\}
\quad
\begin{array}{rl}
\text{log. } a = & 2{\cdot}303196 \\
6 \text{ log. } c = & 13{\cdot}844976 \\
\text{log. } e = & \underline{2{\cdot}311754} \\
 & 18{\cdot}459926 = a + 6c + e \\
 & \underline{9{\cdot}221404} = 4 \text{ log. } b \\
4 & \underline{)9{\cdot}238522} \\
\text{Log. } d \text{ or log. } 204 = & 2{\cdot}309630
\end{array}
$$

2. Given the cube roots of 45, 46, 47, 48, and 49, respectively equal to 3·556893, 3·583048, 3·608826, 3·634241, and 3·659306, to find the cube root of 50. Ans. 3·684033.

3. Given the logarithms of 60, 61, 62, 64, 65, and 66, to find that of 63. Ans. 1·799341.

4. Given the logarithms of 101, 102, 104, and 105, to find that of 103. Ans. 2·012837.

REVERSION OF SERIES.

When an equation is given of this form, $x = az + bz^2 + cz^3 + dz^4 + \&c.$, and it is required to find z in terms of x, the method of doing this is called the Reversion of the Series.

---

* It is obvious that this table is composed of the first terms of the 1st, 2d, 3d, &c. nth orders of differences; and when any of these orders become $= 0$, any intermediate term may be accurately found; but if the differences do not vanish, the result is only an approximation which will come nearer the truth the more terms there are in the given series.

Assume the equation $z = Ax + Bx^2 + Cx^3 + Dx^4 +$ &c., substitute this series and its powers instead of z and its powers in the given equation, then make the coefficients of the like powers of x each $= 0$, and they will give equations for finding the values of A, B, C, D, &c.

Let $x = v + \frac{1}{6}v^3 + \frac{3}{40}v^5 + \frac{15}{336}v^7 + \frac{105}{3456}v^9 +$ &c., and let it be required to find v in terms of x.

Here the assumed equation is $v = Ax + Bx^3 + Cx^5 + Dx^7 + Ex^9 +$ &c. Therefore,

$$\frac{1}{6}v^3 = +\frac{1}{6}x^3 + \frac{3}{6}Bx^5 + \frac{B^2+C}{2}x^7 + \left(\frac{1}{2}D + AB + A^3\right)x^9, \text{ \&c.}$$

$$\frac{3}{40}v^5 = \quad + \frac{3}{40}x^5 + \frac{15}{40}Bx^7 \quad + \left(\frac{3}{4}B^2 + \frac{3}{8}C\right)x^9, \text{ \&c.}$$

$$\frac{15}{336}v^7 = \quad + \frac{15}{336}x^7 \quad + \frac{5}{16}Bx^9, \text{ \&c.}$$

$$\frac{105}{3456}v^9 = \quad + \frac{105}{3456}x^9, \text{ \&c.}$$

And equating the coefficients of the like powers of x, we have $B + \frac{1}{6} = 0$ or $B = -\frac{1}{6}$, $C + \frac{3}{6}B + \frac{3}{40} = 0$ or $C = +\frac{1}{120}$, $D + \frac{1}{2}B^2 + \frac{1}{2}C + \frac{3}{8}B + \frac{5}{112} = 0$ or $D = -\frac{1}{5040}$, &c. Therefore $v = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 +$ &c. $= x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2.3.4.5.6.7} +$ &c., where the law of continuation is evident.

REVERT THE FOLLOWING SERIES.

1. $x = y - y^2 + y^3 - y^4 +$ &c.

   Ans. $y = x + x^2 + x^3 + x^4 +$ &c.

2. $x = y + \frac{1}{2}y^2 + \frac{1}{3}y^3 + \frac{1}{4}y^4 +$ &c.

   Ans. $y = x - \frac{x^2}{2} + \frac{x^3}{2.3} - \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5} -$ &c.

3. $x = \frac{y}{a} - \frac{y^2}{2a^2} + \frac{y^3}{3a^3} - \frac{y^4}{4a^4} +$ &c.

   Ans. $y = a \times \left(x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \text{\&c.}\right)$

4. $x = y - \frac{y^3}{2.3a^2} + \frac{y^5}{2.3.4a^4} - \frac{y^7}{2.3.4.5.6a^6} +$ &c.

   Ans. $y = x + \frac{x^3}{2.3a^2} + \frac{x^5}{2.3.4a^4} + \frac{x^7}{2.3.4.5.6a^6} +$ &c.

5. $x = \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} +$ &c. (put $v = 2x$).

Ans. $y = v^{\frac{1}{2}} - \frac{v}{3} + \frac{v^{\frac{3}{2}}}{36} - \frac{v^2}{170} +$ &c.

6. $x = y^{-\frac{1}{2}} - \frac{y^{\frac{1}{2}}}{2} - \frac{y^{\frac{3}{2}}}{8} - \frac{y^{\frac{5}{2}}}{16} - \frac{y^{\frac{7}{2}}}{121} -$ &c.

Ans. $y = x^{-2} - x^{-4} + x^{-6} - x^{-8} +$ &c.

OF LOGARITHMS.

LOGARITHMS are a set of artificial numbers invented and formed into tables for the purpose of facilitating arithmetical computations. They are adapted to the natural numbers in such a manner, that, by their aid, Addition supplies the place of Multiplication, Subtraction that of Division, Multiplication that of Involution, and Division that of the Extraction of Roots.

Logarithms may be considered as the exponents of the powers to which a given number must be raised, in order to produce all the natural numbers.

Thus, let r be any given number, and let such values be successively assigned to x as will make $r^x = a$, $r^{x'} = b$, $r^{x''} = c$, &c.; then x, x', x'', &c. are the logarithms of a, b, c, &c. respectively.

If $x = 0$, then $r^x = 1$, whatever be the value of r; hence in every system of logarithms the logarithm of 1 is $= 0$. Hence, also, when $x = 1$, it is obvious a will be equal to r. The constant quantity r is called the *radix* or *base* of the system, and in every system it is that number whose logarithm is 1.

Since r may be assumed of any value greater or less than unity, it is evident that there may be innumerable systems of logarithms answering to the natural numbers; but since 10 is the *base* of our system of arithmetic, it has accordingly been assumed as the *base* of our common tables of logarithms; therefore,

Let $r = 10$, and we have $10^{-3} = \frac{1}{1000}$, $10^{-2} = \frac{1}{100}$, $10^{-1} = \frac{1}{10}$, $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, &c. that is, the log. of $\frac{1}{1000}$ or ·001 is -3, of $\frac{1}{100}$ or ·01 is -2, of $\frac{1}{10}$ or ·1 is -1, of 1 is 0, of 10 is 1, of 100 is 2, of 1000 is 3, &c. Hence it is evident that the logarithm of any number falling

between ·001 and ·01 will be $-3+$ some fraction; that of a number between ·01 and ·1 will be $-2+$ some fraction; that of a number between ·1 and 1 will be $-1+$ some fraction; that of a number between 1 and 10 will be a proper fraction; that of a number between 10 and 100 will be $1+$ some fraction; that of a number between 100 and 1000 will be $2+$some fraction, and so on.

It is therefore manifest that in this system the logarithm of any number, and that of another 10, 100, 1000, &c. times greater or less, consist of the same decimal fraction, and differ only in the integral part; so that all numbers, whether they are integers, decimals, or partly integral and partly decimal, have the same positive quantity for the decimal part of their logarithm: Thus,

The logarithm of	2746	is 3·438701
. . . .	274·6	is 2·438701
. . . .	27·46	is 1·438701
. . . .	2·746	is 0·438701
. . . .	·2746	is $\bar{1}$·438701
. . . .	·02746	is $\bar{2}$·438701
. . . .	·002746	is $\bar{3}$·438701.*

PROPERTIES OF LOGARITHMS.

1. Let a and b be any two numbers, and let $r^x=a$, and $r^{x'}=b$; then x is the log. of a, and x' that of b. Now $a\times b = r^x\times r^{x'}=r^{x+x'}$, but the log. of $r^{x+x'}$ is $x+x'$ $\therefore$ the log. of $ab=x+x'=$ log. $a+$log. b. In like manner it may be shown that log. $abc=$ log. $a+$log. $b+$log. c. Hence the logarithm of the product of any number of quantities is equal to the sum of their logarithms.

2. Again, $\frac{a}{b}=\frac{r^x}{r^{x'}}=r^{x-x'}$; but the log. of $r^{x-x'}=x-x'$ $\therefore$ the log. of $\frac{a}{b}=x-x'=$ log. $a-$log. b; hence the logarithm of the quotient of any two numbers is equal to the difference of the logarithms of these numbers, or the log. of a fraction $\frac{a}{b}$ is equal to the log. of the numerator minus that of its denominator. If a is less than b, then log. $a-$log. b is negative; consequently the logarithms of all proper fractions are negative.

* When the index of the logarithm is negative, the sign — is generally put above it in order to distinguish it from the decimal part, which must always be considered as + or affirmative.

3. Let $a = r^x$ be raised to the nth power, then $a^n = r^{xn}$; but the log. of r^{xn} is xn $\therefore$ the log. of $a^n = xn = n$ times the log. of a. In like manner, taking the nth root of $a = r^x$, we have $a^{\frac{1}{n}} = r^{\frac{x}{n}}$; but the log. of $r^{\frac{x}{n}}$ is $\frac{x}{n}$ $\therefore$ the log. of $a^{\frac{1}{n}} = \frac{x}{n}$ $= \frac{\text{log. } a}{n}$; hence the logarithm of the nth power of any number is equal to its logarithm multiplied by n, and the logarithm of the nth root of any number is equal to its logarithm divided by n.

4. Let a, na, n^2a, n^3a, &c. be a series of numbers in geometrical progression, such that x is the log. of a, and y that of n; then $r^x = a$, and $r^y = n$; and the logarithms of the numbers in the geometrical progression will be r^x, r^{x+y}, r^{x+2y}, r^{x+3y}, &c., which evidently form an arithmetical progression. Hence, if a series of quantities are in Geometrical Progression, their logarithms are in Arithmetical Progression.

These principles being of the most extensive use in algebraical calculations, the following examples are given as exercises to the student:—

1. Log. $(a.b.c.d. \ldots) = \text{log. } a + \text{log. } b + \text{log. } c + \text{log. } d \ldots$

2. Log. $\left(\frac{abc}{de}\right) = \text{log. } a + \text{log. } b + \text{log. } c - \text{log. } d - \text{log. } e$.

3. Log. $(a^m.b^n.c^v) = m. \text{ log. } a + n \text{ log. } b + v \text{ log. } c$.

4. Log. $\left(\frac{a^m b^n}{c^v d^s}\right) = m \text{ log. } a + n \text{ log. } b - v \text{ log. } c - s \text{ log. } d$.

5. Log. $(a^2 - b^2) = \text{log. } (a+b)(a-b) = \text{log. } (a+b) + \text{log. } (a-b)$.

6. Log. $(a^2 - b^2)^{\frac{1}{2}} = \frac{1}{2} \text{ log. } (a+b) + \frac{1}{2} \text{ log. } (a-b)$.

7. Log. $(a^3.a^{\frac{3}{4}}) = \text{log. } a^3 + \frac{3}{4} \text{ log. } a = 3 \text{ log. } a + \frac{3}{4} \text{ log. } a = \frac{15}{4}$ log. a.

8. Log. $(a^3 - b^3)^{\frac{m}{n}} = \frac{m}{n} \text{ log. } (a-b) + \frac{m}{n} \text{ log. } (a^2 + ab + b^2)$, or making $(z^2 = ab) = \frac{m}{n}\{\text{log. } (a-b) + \text{log. } (a+b+z) + \text{log. } (a+b-z)\}$.

9. Log. $(a^2 + b^2)^{\frac{1}{2}}$, making $2ab = z^2$, it becomes $\text{log.}\{(a+b)^2 - z^2\}^{\frac{1}{2}} = \frac{1}{2}\{\text{log. } (a+b+z) + \text{log. } (a+b-z)\}$.

10. Log. $\frac{(a^2 - b^2)^{\frac{1}{2}}}{(a+b)^2} = \frac{1}{2}\{\text{log. } (a-b) - 3 \text{ log. } (a+b)\}$.

11. To insert m geometric means between a and y. In the equation $y = ar^{n-1}$, page 75, Prop. I., let $n = m+2$; then the ratio $r = \left(\frac{y}{a}\right)^{\frac{1}{m+1}}$, and log. $r = \frac{\text{log. } y - \text{log. } a}{m+1}$; hence the several means are ar, ar^2, ar^3, &c. . . ar^m, and their logs. are log. a + log. r, log. a + 2 log. r, log. a + 3 log. r, &c. log. $a + m$ log. r.

Let it be required to insert 10 means between 1 and 2; here log. $a = 0$, and log. $r = \frac{1}{11}$ log. $2 = 0{\cdot}02736636$; hence $r = 1{\cdot}065041$, and the logarithms of the consecutive terms are 2 log. r, 3 log. r, 4 log. r, &c. The progression is therefore 1, 1·065041, 1·134312, 1·208089, 1·286665, 1·370351, 1·459480, 1·554406, 1·655506, 1·763182, 1·877862, 2.

PROB. I. To find the logarithm of any given number.

Let $r^x = \text{N}$, then if x be found in terms of r and N, it will be the logarithm of N to the *base* r.

Put $\text{N} = 1 + n$, and $r = 1 + a$; then $(1+a)^x = 1+n$; and, raising both to the mth power, we have $(1+a)^{xm} = (1+n)^m$, whatever be the value of m. Expand both sides of this equation and it becomes

$$1 + xma + \frac{xm(xm-1)}{2}a^2 + \frac{xm(xm-1)(xm-2)}{2.3}a^3 + \text{\&c.} =$$

$$1 + mn + \frac{m(m-1)}{2}n^2 + \frac{m(m-1)(m-2)}{2.3}n^3 + \text{\&c.}$$

Now, expunging 1 from both sides of the equation, and dividing by m, we obtain

$$x\left(a + \frac{xm-1}{2}a^2 + \frac{(xm-1)(xm-2)}{2.3}a^3 + \text{\&c.}\right) = n + \frac{m-1}{2}n^2$$

$+ \frac{(m-1)(m-2)}{2.3}n^3 +$ &c.; and since m may be of any value, let us suppose $m = 0$, and the equation becomes

$$x(a - \tfrac{1}{2}a^2 + \tfrac{1}{3}a^3 - \tfrac{1}{4}a^4 + \text{\&c.}) = n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 + \text{\&c.}$$

$$\therefore \text{ the log. of } (1+n) = x = \frac{n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \text{\&c.}}{a - \frac{1}{2}a^2 + \frac{1}{3}a^3 - \frac{1}{4}a^4 + \text{\&c.}}$$

Substituting in this equation for n and a their values $\text{N}-1$ and $r-1$, we obtain

$$\text{Log. N} = \frac{(\text{N}-1) - \frac{1}{2}(\text{N}-1)^2 + \frac{1}{3}(\text{N}-1)^3 - \frac{1}{4}(\text{N}-1)^4 + \text{\&c.}}{(r-1) - \frac{1}{2}(r-1)^2 + \frac{1}{3}(r-1)^3 - \frac{1}{4}(r-1)^4 + \text{\&c.}}$$

$$= \text{M}\{(\text{N}-1) - \tfrac{1}{2}(\text{N}-1)^2 + \tfrac{1}{3}(\text{N}-1)^3 - \tfrac{1}{4}(\text{N}-1)^4 + \text{\&c.}\}$$

where M is the *modulus* of the system

$$= \frac{1}{(r-1) - \frac{1}{2}(r-1)^2 + \frac{1}{3}(r-1)^3 - \frac{1}{4}(r-1)^4 + \text{\&c.}}$$

This series, therefore, gives us the value of x in terms of r and N; but if N be any number greater than unity, it is evidently a diverging series, and of little use in the construction of logarithms.

In order to obtain a converging series, let us suppose $N = n - 1$; and, resuming the equation,

Log. $(1+n = \frac{n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \&c.}{a - \frac{1}{2}a^2 + \frac{1}{3}a^3 - \frac{1}{4}a^4 + \&c.} = M(n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \&c.)$, proceeding as before, we get log. $(1-n) = M(-n - \frac{1}{2}n^2 - \frac{1}{3}n^3 - \frac{1}{4}n^4 - \&c.)$; and, subtracting this from the former, we obtain log. $(1+n)$ — log. $(1-n)$ = log. $\frac{1+n}{1-n} = 2M(n + \frac{1}{3}n^3 + \frac{1}{5}n^5 + \frac{1}{7}n^7 + \&c.)$; and as this equation is true for every value of n,

Let $n = \frac{1}{N-1}$, then $\frac{1+n}{1-n} = \frac{N}{N-2}$, and consequently log. $\frac{N}{N-2} = 2M\left\{\frac{1}{N-1} + \frac{1}{3(N-1)^3} + \frac{1}{5(N-1)^5} + \frac{1}{7(N-1)^7} + \&c.\right\}$; hence log. $N = 2M\left\{\frac{1}{N-1} + \frac{1}{3(N-1)^3} + \frac{1}{5(N-1)^5} + \frac{1}{7(N-1)^7} + \&c.\right\}$ + log. $(N-2)$, which is a series rapidly convergent, and therefore very convenient for the construction of logarithms.

Before proceeding farther, however, it is necessary to assign some particular value to M; and since its value is arbitrary, let it be $= 1$, or the value first assumed by Lord Napier. Then log. $N = 2\left\{\frac{1}{N-1} + \frac{1}{3(N-1)^3} + \frac{1}{5(N-1)^5} + \frac{1}{7(N-1)^7} + \&c.\right\}$ + log. $(N-2)$; but it is obvious that N must be some number greater than 2, and we must therefore first find the log. of 2, which may be done, by supposing $N = 4$. Hence log. 4 = log. 2^2 = 2 log. $2 = 2\left(\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \&c.\right)$ + log. 2; and, expunging log. 2 from each side of the equation, it becomes log. $2 = 2\left(\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \frac{1}{9.3^9} + \frac{1}{11.3^{11}} + \frac{1}{13.3^{13}}\right) = 0{\cdot}6931472$. Having thus found log. 2, and, availing ourselves of the properties of logarithms, we

readily obtain the logarithms of all numbers, by substituting in the formula 1, 2, 3, &c. for N: Thus,

$$
\begin{aligned}
\text{Log. } 1 &= \quad \ldots\ldots \quad &&= 0\cdot0000000 \\
2 &= \quad \ldots\ldots \quad &&= 0\cdot6931472 \\
3 &= 2\left(\frac{1}{2}+\frac{1}{3.2^3}+\frac{1}{5.2^5}+\frac{1}{7.2^7}+\frac{1}{9.2^9}+\&c.\right) &&= 1\cdot0986123 \\
4 &= \log.\ 2^2 = 2\log.\ 2 \quad \ldots &&= 1\cdot3862944 \\
5 &= 2\left(\frac{1}{4}+\frac{1}{3.4^3}+\frac{1}{5.4^5}+\frac{1}{7.4^7}+\frac{1}{9.4^9}+\&c.\right) && \\
&\quad + \log.\ 3 \quad \ldots &&= 1\cdot6094379 \\
6 &= \log.\ 3+\log.\ 2 \quad \ldots &&= 1\cdot7917595 \\
7 &= 2\left(\frac{1}{6}+\frac{1}{3.6^3}+\frac{1}{5.6^5}+\frac{1}{7.6^7}\right)+\log.\ 5 &&= 1\cdot9459101 \\
8 &= \log.\ 2^3 = 3\log.\ 2 \quad \ldots &&= 2\cdot0794415 \\
9 &= \log.\ 3^2 = 2\log.\ 3 \quad \ldots &&= 2\cdot1972246 \\
10 &= \log.\ 2+\log.\ 5 \quad \ldots &&= 2\cdot3025851 \\
\&c. &= \quad \&c. \quad \ldots &&= \quad \&c.
\end{aligned}
$$

These are called Napierean Logarithms, from the name of their ingenious inventor; but they are likewise commonly known by the name of Hyperbolic Logarithms, from their connexion with the quadrature of the hyperbola.

Prob. II. To find the value of the *base* r in this system, whose *modulus* is 1.

Let log. N or log. $(n+1)=l$, and since $M=1$, $l=n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\&c.$; reverting this series, we have $1+n$ or $N=1+l+\frac{l^2}{2}+\frac{l^3}{2.3}+\frac{l^4}{2.3.4}+\&c.$; and since the base of any system of logarithms is that number whose log. is 1, let $l=1$, and we obtain the base $r=1+1+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}+\&c.=2\cdot7182818.$

Since log. $2\cdot7182818=1$

$$
\begin{aligned}
\therefore\ (2\cdot7182818)^{\cdot6931472} &\text{ or log. } (2\cdot7182818)^2 = 2 \\
(2\cdot7182818)^{1\cdot0986123} &\text{ or log. } (2\cdot7182818)^3 = 3 \\
\&c. \qquad & \qquad \&c.
\end{aligned}
$$

Hence the numbers whose hyperbolic logarithms are 1, 2, 3, 4, &c. are *decimal* numbers, and therefore inconvenient for ordinary arithmetical computations.

Prob. III. Having given the logarithm of any number to the *base* r, to find its logarithm to any other base s.

Let $s^x = a$, or $x = \text{log. } a$ to the base s; then, using the given logs. to the base r, we have $x = \frac{\text{log. } a}{\text{log. } s} = \text{log. } a \times \frac{1}{\text{log. } s}$; hence the log. of a to the base s is found by multiplying the log. of a to the base r by the reciprocal of the log. of s to the base r. But this reciprocal is the *modulus* of the system to which it belongs, and is therefore constant in that system; consequently if we suppose $s = 10$, the base of our common logarithms, we will obtain the modulus of these logarithms, or $\text{M} = \frac{1}{\text{hyp. log. } 10} = \frac{1}{2{\cdot}3025851} = {\cdot}4342944819$, and $2\text{M} = {\cdot}8685889638$. To construct a table of common logarithms, therefore, we have only to multiply the hyperbolic logarithms already found by this value of M, or, which is the same thing, substitute the value of 2M in the formula whence we derived the hyperbolic logarithms. Thus the common log. of N is $\cdot 86858896 \left\{\frac{1}{\text{N}-1} + \frac{1}{3(\text{N}-1)^3} + \frac{1}{5(\text{N}-1)^5} + \text{\&c.}\right\} + \text{log. } (\text{N}-2)$.

Now, making N successively 1, 2, 3, 4, &c. and availing ourselves of the properties of logarithms as before, we have

$$
\begin{array}{lll}
\text{Log. } 1 = & \ldots & = 0{\cdot}0000000 \\
2 = & \cdot 86858896\left(\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \text{\&c. to 7 terms}\right) & = 0{\cdot}3010300 \\
3 = & \cdot 86858896\left(\frac{1}{2} + \frac{1}{3.2^3} + \frac{1}{5.2^5} + \text{\&c. to 10 terms}\right) & = 0{\cdot}4771213 \\
4 = & \text{log. } 2^2 = 2 \text{ log. } 2 & = 0{\cdot}6020600 \\
5 = & \text{log. } \frac{10}{2} = \text{log. } 10 - \text{log. } 2 & = 0{\cdot}6989700 \\
6 = & \text{log. } 3 + \text{log. } 2 & = 0{\cdot}7781513 \\
7 = & \cdot 86858896\left(\frac{1}{6} + \frac{1}{3.6^3} + \frac{1}{5.6^5} + \frac{1}{7.6^7}\right) + \text{log. } 5 & = 0{\cdot}8450980 \\
8 = & \text{log. } 2^3 = 3 \text{ log. } 2 & = 0{\cdot}9030900 \\
9 = & \text{log. } 3^2 = 2 \text{ log. } 3 & = 0{\cdot}9542425 \\
10 = & \ldots & = 1{\cdot}0000000 \\
\text{\&c.} = & \text{\&c.} & = \text{\&c.}
\end{array}
$$

As exercises, the student may calculate the hyperbolic and also the common logarithms of the numbers from 10 to 20.

NOTE. To obtain the logarithms true to 7 places of decimals, *three* terms of the series will be sufficient for numbers between 10 and 29 inclusive; *two* terms for numbers between 29 and 400; and *one* term for all numbers above 400.

APPLICATION OF LOGARITHMS.

THE index or integral part of the logarithm of any whole or mixed number, as has already been shown, is always *one less than the number of integral figures* of which that number consists; and, in decimal fractions, the index, which is negative, is that number which points out *the distance of the first significant figure from the place of units.* Instead of negative indices, their *arithmetical complements to* 10 are often used. Thus if there is no cipher between the decimal point and the first significant figure of the decimal, the index is $\bar{1}$ or 9; if there is one cipher between them, the index is $\bar{2}$ or 8; if two ciphers are between them, it is $\bar{3}$ or 7, and so on.

The indices being thus readily found are omitted in the common logarithmic tables, and the decimal part only of the logarithms inserted.

TO FIND THE LOGARITHM OF A NUMBER FROM THE TABLES.

Look for the three highest figures in the margin on the left hand, and running along that line to the column which has the fourth figure at the top, you will find the logarithm for these four figures. If the number consists of more than four figures, take the difference between the logarithm thus found and the next greater, and multiply it by the remaining figures, and from the product cut off as many figures as are in the multiplier; the rest added to the logarithm for the first four figures gives the logarithm required.

NOTE. The mean differences given under D in the right-hand column may be used, except in the first three pages of the table, where they vary rapidly.

1. Required the logarithm of 73284.

Look in the margin for 732, and on that line in the column which has 8 at the top you will find ·864985, the logarithm of 7328, and the difference between it and the next logarithm is 60, which, multiplied by 4, gives 240: therefore, adding 24 to ·864985, we have 4·865009 for the logarithm of 73284, with 4 for an index, because the number has five places. If the number had been 732·84, the logarithm would have been the same, but the index would have been 2.

2. Required the log. of 6·1953. . . Ans. 0·792062.
3. . . . . of 47·5384. . . . 1·677044.
4. . . . . of ·003825. . . $\bar{3}$·582631.

TO FIND THE NUMBER CORRESPONDING TO A GIVEN LOGARITHM.

If the given logarithm be found in the table, the first three figures of the number will be found on the same line in the margin, and the

fourth at the top of the column. But if the logarithm be not found exactly in the table, take the number answering the next less, and subtract this logarithm from the given one, and also from the next greater in the table; and, annexing ciphers to the first remainder, divide it by the other, to get the fifth, sixth, &c. figures. The integer places must be one more than the units in the index, and the rest are decimals.

5. Required the number corresponding to the logarithm 4·597179.

The next less logarithm is ·597146, and the number answering to it is 3955; the difference between it and the given logarithm is 33, and between it and the next greater in the table is 110. Divide 330 by 110, and the quotient 3, annexed to 3955, gives 39553 for the number sought.

6. Required the number of log.	3·774240. .	Ans. 5946·2.
7.	2·147522. .	140·45.
8.	$\bar{2}$·862489. . .	0·07286.

TO FIND THE ARITHMETICAL COMPLEMENT.

Subtract the logarithm from 10, an integer, or subtract the right-hand figure from 10, and all the rest from 9.

9. Thus the arithmetical complement of 3·642754 is 6·357246.

10. Required the ar. co. of	2·749367.	Ans. 7·250633.
11. . . . . of	1·360797. .	8·639203.

TO PERFORM MULTIPLICATION BY LOGARITHMS.

Add the logarithms of the factors; the sum is the logarithm of the product.

NOTE. A negative index must be subtracted when the logarithm is added, and added when the logarithm is subtracted.

12. Multiply	37·68	log. 1·576111
by	9·25	log. 0·966142
Product	348·54	log. 2·542253

13. Multiply 5·735, 0·023, and 56·25 together.

	5·735	log. 0·758533
	0·023	log. $\bar{2}$·361728
	56·25	log. 1·750123
Product	7·41966	log. 0·870384

14. Required the product of	7·542 by ·963.	Ans. 7·26295.
15.	·00352 by ·864. .	0·0030413.
16.	·0925 by 73·5. .	6·79875.

TO PERFORM DIVISION BY LOGARITHMS.

Subtract the logarithm of the divisor from that of the dividend: the remainder is the logarithm of the quotient.

Or add the arithmetical complement of the divisor to the logarithm of the dividend: the sum, with its index lessened by 10, is the logarithm of the quotient.

17. Divide	9·7128	log. 0·987344	log.	0·987344
by	0·456	log. 9·658965	ar. co.	0·341035
Quotient	21·3	log. 1·328379	log.	1·328379

18. Required the quotient of 9 by 75. . Ans. 0·12.
19. . . . . 8964 by 3·84. . 2334·376.
20. . . . . 62·78 by 71·6. . ·876814.

TO WORK PROPORTION BY LOGARITHMS.

Add the logarithms of the second and third terms together, and from their sum subtract the logarithm of the first: the remainder is the logarithm of the fourth term, or answer.

Or add together the arithmetical complement of the first term, and the logarithms of the other two: the sum, with its index lessened by 10, is the logarithm of the answer.

21. First	36	log. 1·556303	ar. co.	8·443697
Second	144	log. 2·158362	log.	2·158362
Third	28	log. 1·447158	log.	1·447158
		3·605520		
Fourth	112	log. 2·049217	log.	2·049217

22. If 17 men do a piece of work in 28 days, in what time will 12 do it? Ans. 39·66667 or 39⅔ days.

23. If 13¼ cwt. be carried 57 miles for £2·568, how far should 34½ cwt. be carried for £8·56? Ans. 72·97116 miles.

TO INVOLVE A NUMBER BY LOGARITHMS.

Multiply the logarithm by the name of the power: the product is the logarithm of the power.

24. Numb. 32 log. 1·505150		Numb. ·009 log. $\bar{3}$·954243
3		3
3d power 32768 log. 4·515450		·000000729 log. $\bar{7}$·862729

NOTE. After multiplying the negative index, the carriage to it from the logarithm must be subtracted from the product. If the positive index be used, 10 times the name of the power lessened by 1 must be taken from the index of the power.

25. Number ·0437	log. $\bar{2}$·640481, or	8·640481
	4	4
4th power ·000003649	log. $\bar{6}$·561924	4·561924

TO EXTRACT THE ROOT OF A NUMBER BY LOGARITHMS.

Divide its logarithm by the name of the root: the quotient is the logarithm of the root.

NOTE. If the given number be a decimal, and its index positive, prefix the name of the root lessened by 1 to the index, before dividing. If the index be negative, add to it the least number that will make the sum divisible by the name of the root: the quotient is the index of the root; but in dividing the logarithm, the number added only is to be considered as the index.

26. Number ·00130321 log. $4\,|\,\bar{3}{\cdot}115014$ log. $37{\cdot}115014$
1

Fourth root ·19 log. $\bar{1}{\cdot}278754$ log. $9{\cdot}278754$

27. Number 9261 log. $3)\underline{3{\cdot}966658}$
Cube root 21 log. $1{\cdot}322219$

28. Required the square root of ·5329. . Ans. ·73.
29. . . . . cube root of ·041063625. . ·345.
30. . . . . fourth root of 7. . . 1·626567.

EXERCISES.

1. Req[d]. the seventh power of 7·142. Ans. 947850.
2. . . . sixth root of 2. 1·1224625.
3. . . . ninth power of ·0375. ·0000000000001466.
4. . . . eighth root of ·02405. ·627536.
5. . . . compound interest of £67·495 for $5\frac{1}{3}$ years, at 4 per cent. Ans. £15·7033 = £15, 14s. $0\frac{3}{4}$d.
6. . . . rate of comp. int. at which £136·782 will, in $5\frac{1}{4}$ years, amount to £173·564. Ans. 4·64.
7. . . . time in which £53·5 will amount to £76·36, at $3\frac{1}{2}$ per cent. comp. int. Ans. 10·342 years.

SOLUTION OF QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

Quadratic Equations, containing two unknown quantities, when in their most complete form, are expressed thus:

$$ax^2 + by^2 + cxy + dx + ey = m$$
$$a'x^2 + b'y^2 + c'xy + d'x + e'y = m'.$$

The general solution of these equations can only be obtained by means of equations of higher dimensions than quadratics. There are, however, particular cases which admit of solution by the rules formerly given for the solution of quadratics with only one unknown quantity.

Case I. When one of the equations is a simple equation.

Find the value of one of the unknown quantities from the simple equation, and substitute for it this value in the other equation; the resulting equation will be a quadratic containing only one unknown quantity.

Let the equations be $\left.\begin{array}{r} 3x + 2y = 13 \\ \text{and } x^2 + 3xy - y^2 = 23 \end{array}\right\}$ To find the values of x and y.

From the first equation $x = \dfrac{13 - 2y}{3}$, $\because x^2 = \dfrac{169 - 52y + 4y^2}{9}$

Substituting these values for x and x^2 in the second equation, we have

$$\frac{169 - 52y + 4y^2}{9} + 13y - 2y^2 - y^2 = 23$$

$$\text{Or } 23y^2 - 65y = -38$$

Dividing by 23, $\quad y^2 - \dfrac{65y}{23} = -\dfrac{38}{23}$

Complete the square, $y^2 - \dfrac{65y}{23} + \dfrac{4225}{2116} = \dfrac{4225}{2116} - \dfrac{3496}{2116} = \dfrac{729}{2116}$.

Extract the root, $\quad y - \dfrac{65}{46} = \pm \dfrac{27}{46}$;

$$\text{Hence} \quad y = \frac{65}{45} \pm \frac{27}{46} = \frac{92}{46}, \text{ or } \frac{38}{46} = 2, \text{ or } \frac{19}{23}.$$

$$\therefore\ x = \frac{13 - 2y}{3} = \frac{13 - 4}{3}, \text{ or } \frac{13 - \frac{38}{23}}{3} = 3, \text{ or } 3\tfrac{18}{23}.$$

Let the equations be $\left.\begin{array}{r}\frac{x+2y}{3} = 9 \\ \text{and } 4xy = 280\end{array}\right\}$ To find the values of x and y.

From the first equation we have

$$x + 2y = 27 \text{ or } x = 27 - 2y.$$

Substituting this value for x in the second equation, we have

$$108y - 8y^2 = 280$$

Or $$8y^2 - 108y = -280$$

Divide by 8, $$y^2 - \frac{27y}{2} = -35$$

Complete the square, $$y^2 - \frac{27y}{2} + \frac{729}{16} = \frac{729}{16} - \frac{560}{16} = \frac{169}{16}$$

Extract the root, $$y - \frac{27}{4} = \pm\frac{13}{4}$$

$$\therefore\ y = \frac{27}{4} \pm \frac{13}{4} = 10, \text{ or } 3\tfrac{1}{2}$$

Whence $x = 27 - 2y = 27 - 20$, or $27 - 7 = 7$, or 20.

Let the equations be $\left.\begin{array}{r}x + 2y = 16 \\ \text{and } 2x^2 + xy + 4y^2 = 152\end{array}\right\}$ To find the values of x and y.

From the first equation $x = 16 - 2y$, and $x^2 = 256 - 64y + 4y^2$.

Substituting these values of x and x^2 in the second equation, we have $$512 - 128y + 8y^2 - 16y + 2y^2 + 4y^2 = 152.$$

Or $$14y^2 - 144y = 152 - 512 = -360$$

Divide by 14, $$y^2 - \frac{72}{7}y = -\frac{180}{7}$$

Complete the square, $$y^2 - \frac{72}{7}y + \frac{1296}{49} = \frac{1296}{49} - \frac{1260}{49} = \frac{36}{49}$$

Extract the root, $$y - \frac{36}{7} = \pm\frac{6}{7}$$

$$\therefore\ y = \frac{36}{7} \pm \frac{6}{7} = 6, \text{ or } 4\tfrac{2}{7}$$

Whence $x = 16 - 2y = 16 - 12$, or $16 - 8\tfrac{4}{7} = 4$, or $7\tfrac{3}{7}$.

EQUATIONS.	ANSWERS.
1. $\left.\begin{array}{r}x - y = 2 \\ x^2 + y^2 = 20\end{array}\right\}$	$\left\{\begin{array}{l}x = 4 \\ y = 2.\end{array}\right.$

2. $\left.\begin{aligned} x + y &= 6 \\ x^2 - y^2 &= 12 \end{aligned}\right\}$ $\quad\left\{\begin{aligned} x &= 4 \\ y &= 2. \end{aligned}\right.$

3. $\left.\begin{aligned} 3x + y &= 42 \\ \frac{xy}{6} + 2y^2 &= 308 \end{aligned}\right\}$ $\quad\left\{\begin{aligned} x &= 10 \\ y &= 12. \end{aligned}\right.$

4. $\left.\begin{aligned} x + 5y &= 39 \\ 2x^2 - 3xy + y^2 &= 36 \end{aligned}\right\}$ $\quad\left\{\begin{aligned} x &= 9 \\ y &= 6. \end{aligned}\right.$

5. $\left.\begin{aligned} 2x - 3y &= -11 \\ x^2 + x + 2y^2 - xy &= 93 \end{aligned}\right\}$ $\quad\left\{\begin{aligned} x &= 5 \\ y &= 7. \end{aligned}\right.$

6. $\begin{aligned} x+y : x - y &:: 7 : 3 \\ x - y^2 &= 1 \end{aligned}$ $\quad\left\{\begin{aligned} x &= 5 \\ y &= 2. \end{aligned}\right.$

Case II. When x^2, y^2, or xy, is found in every term of the two equations, that is, when they are of the form

$$ax^2 + bxy + cy^2 = d$$
$$a'x^2 + b'xy + c'y^2 = d'.$$

Assume $x = vy$, then $x^2 = v^2y^2$, and substitute these values of x and x^2 in both equations, then find the value of y^2 in both equations, and make these equal to each other, and we obtain a quadratic, whence the value of v may be found.

Let the equations be $\left.\begin{aligned} x^2 + xy &= 40 \\ \text{and} \quad 3xy - 2y^2 &= 27 \end{aligned}\right\}$ To find the values of x and y.

Assume $x = vy$ and $x^2 = v^2y^2$, substitute these values of x and x^2 in the given equations, and we have

$$v^2 y^2 + vy^2 = 40, \text{ or } y^2 = \frac{40}{v^2 + v}.$$

$$\text{and} \qquad 3vy^2 - 2y^2 = 27, \text{ or } y^2 = \frac{27}{3v - 2}.$$

Hence equating these values of y^2, we have

$$\frac{40}{v^2 + v} = \frac{27}{3v - 2}$$

$$\text{or} \quad 120v - 80 = 27v^2 + 27v$$

$$\text{that is,} \quad 27v^2 - 93v = -80$$

Divide by 27, $\quad v^2 - \frac{31}{9}v = -\frac{80}{27}.$

Complete the square, $v^2 - \frac{31}{9}v + \frac{961}{324} = \frac{961}{324} - \frac{960}{324} = \frac{1}{324}.$

Extract the root, $\quad v - \frac{31}{18} = \pm \frac{1}{18}$

$$\therefore \quad v = \frac{31}{18} \pm \frac{1}{18} = \frac{32}{18}, \text{or } \frac{30}{18} = \frac{16}{9}, \text{or } \frac{5}{3}.$$

Whence $y^2 = \frac{40}{v^2 + v} = \frac{40}{\frac{40}{9}}, \text{or } \frac{40}{\frac{400}{81}} = 40 \times \frac{9}{40}, \text{or } 40 \times \frac{81}{400} =$

$$9 \text{ or } \frac{81}{10} \therefore y = \pm 3, \text{or } \pm 9\sqrt{\frac{1}{10}}$$

and $x = vy = \frac{5}{3} \times \pm 3, \text{or } \frac{5}{3} \times \pm 9\sqrt{\frac{1}{10}} = \pm 5, \text{or } \pm 15$ $\sqrt{\frac{1}{10}}$. From which it appears, that, when both of the equations are quadratic, each of the quantities has four values.

Let the equations be $\left.\begin{array}{l} 2x^2 + 2xy + y^2 = 424 \\ \text{and} \quad 2x^2 + 2y^2 = 328 \end{array}\right\}$ To find the values of x and y.

Assume $x = vy$ and $x^2 = v^2 y^2$; substitute these values for x and x^2 in the given equations, and we have

$$2v^2 y^2 + 2vy^2 + y^2 = 424, \text{ or } y^2 = \frac{424}{2v^2 + 2v + 1}$$

$$\text{and } 2v^2 y^2 + 2y^2 = 328, \text{ or } y^2 = \frac{328}{2v^2 + 2}.$$

Hence $\qquad \frac{424}{2v^2 + 2v + 1} = \frac{328}{2v^2 + 2}$

or $\qquad 192v^2 - 656v = -520$

Divide by 192, $\qquad v^2 - \frac{41}{12}v = -\frac{65}{24}$

Complete the square, $v^2 - \frac{41}{12}v + \frac{1681}{576} = \frac{1681}{576} - \frac{1560}{576} = \frac{121}{576}$

Extract the root, $\qquad v - \frac{41}{24} = \pm \frac{11}{24}.$

$$\therefore \quad v = \frac{41}{24} \pm \frac{11}{24} = \frac{52}{24}, \text{or } \frac{30}{24} = \frac{13}{6}, \text{or } \frac{5}{4}$$

Whence $y^2 = \frac{328}{2v^2 + 2} = \frac{328}{\frac{169}{18} + \frac{78}{18}}, \text{or } \frac{328}{\frac{25}{8} + \frac{16}{8}} = 328 \times \frac{18}{247},$

or $328 \times \frac{8}{41} = \frac{5904}{247}, \text{or } 64 \therefore y = \pm 12\sqrt{\frac{41}{247}}, \text{or } \pm 8,$ and

$x = \frac{13}{6} \times \pm 12 \sqrt{\frac{41}{247}} = \pm 26 \sqrt{\frac{41}{247}}$, or $\frac{5}{4} \times \pm 8 =$ ± 10.

EQUATIONS.	ANSWERS.
1. $\left.\begin{array}{l} 2x^2 - y^2 = 46 \\ 3x^2 - 4xy = 35 \end{array}\right\}$	$\left\{\begin{array}{l} x = \pm 5 \\ y = \pm 2. \end{array}\right.$
2. $\left.\begin{array}{l} 3x^2 + y^2 = 4xy + 20 \\ 5x^2 + 2xy = 8y^2 + 64 \end{array}\right\}$	$\left\{\begin{array}{l} x = \pm 4 \\ y = \pm 2. \end{array}\right.$
3. $\left.\begin{array}{l} 4x^2 - 3xy + 6y^2 = 168 \\ 4y^2 - 2x - 3xy = -20 \end{array}\right\}$	$\left\{\begin{array}{l} x = \pm 6 \\ y = \pm 4. \end{array}\right.$
4. $\left.\begin{array}{l} 2y^2 - 3xy + x^2 = 4 \\ -y^2 + 2xy - 3x^2 = -9 \end{array}\right\}$	$\left\{\begin{array}{l} x = \pm 2, \text{ or } \mp \frac{7}{\sqrt{18}} \\ y = \pm 3, \text{ or } \mp \frac{1}{\sqrt{18}} \end{array}\right.$
5. $\left.\begin{array}{l} 7x^2 + x - 5xy = 460 \\ 3x^2 - y - 15y^2 = -80 \end{array}\right\}$	$\left\{\begin{array}{l} x = \pm 10 \\ y = \pm 5. \end{array}\right.$
6. $\left.\begin{array}{l} 3x^2 + 2xy - 4y^2 = 108 \\ -7y^2 - 3xy + x^2 = -81 \end{array}\right\}$	$\left\{\begin{array}{l} x = \pm 6 \\ y = \pm 3. \end{array}\right.$

CASE III. When the equations are symmetrical, or when the two unknown quantities are similarly involved.

The most general form in which such equations are expressed is

$$a\ (x^2 + y^2) + bxy + c\ (x + y) = d$$
$$a'\ (x^2 + y^2) + b'xy + c'\ (x + y) = d'.$$

Substitute for the two unknown quantities the sum and difference of two other unknown quantities, and the result will be an equation containing the additive quantity and its square, and only the square of the subtractive quantity, which, when eliminated, leaves a quadratic containing the additive quantity, whose value is easily found, as well as the original unknown quantities.

Let the equations be $\left.\begin{array}{r} \frac{x}{y} + \frac{y}{x} = 2\frac{1}{2} \\ \text{and } x + xy + y = 27. \end{array}\right\}$ To find the values of x and y.

Assuming $x = v + z$ and $y = v - z$, and substituting these values in both equations, we obtain

$$\frac{v+z}{v-z} + \frac{v-z}{v+z} = 2\frac{1}{2}$$
$$\text{and } v + z + v^2 - z^2 + v - z = 27$$

or by reduction $4v^2 + 4z^2 = 5v^2 - 5z^2$, or $z^2 = \frac{v^2}{9}$ (A)

and $2v + v^2 - z^2 = 27$, or $z^2 = v^2 + 2v - 27$ (B).

Equating these values, we have

$$v^2 + 2v - 27 = \frac{v^2}{9}$$

Multiply by 9, $9v^2 + 18v - 243 = v^2$

Or $8v^2 + 18v = 243$

Divide by 8, $v^2 + \frac{9}{4}v = \frac{243}{8}$

Complete the square, $v^2 + \frac{9}{4}v + \frac{81}{64} = \frac{1944}{64} + \frac{81}{64} = \frac{2025}{64}$.

Extract the root, $v + \frac{9}{8} = \pm\frac{45}{8}$

$$\therefore \quad v = \pm\frac{45}{8} - \frac{9}{8} = \frac{9}{2}, \text{ or } -\frac{27}{4}.$$

Substituting these values of v in equation (A) we have

$$z^2 = \left(\frac{9}{2}\right)^2 \div 9, \text{ or } \left(-\frac{27}{4}\right)^2 \div 9$$

$$\text{that is, } z^2 = \frac{9}{4}, \text{ or } \frac{81}{16}.$$

$$\therefore \quad z = \pm\frac{3}{2}, \text{ or } \pm\frac{9}{4}.$$

Now since $x = v + z$, and $y = v - z$, we have from these values

$$x = \frac{9}{2} \pm \frac{3}{2}, \text{ or } -\frac{27}{4} \pm \frac{9}{4} = 6, 3, -4\tfrac{1}{2}, \text{ or } -9.$$

$$y = \frac{9}{2} \mp \frac{3}{2}, \text{ or } \mp\frac{27}{4} \mp \frac{9}{4} = 3, 6, -9, \text{ or } -4\tfrac{1}{2}.$$

From the manner in which we obtain the values of x and y from the values of v and z, it is manifest that the *first* value of x must always be equal to the *second* value of y, and conversely, and that when v and z have two values, the *third* value of x must be equal to the *fourth* value of y, and conversely. In short, since the nature of these equations is such that x or x^2, and its sign, may always be substituted for y or y^2, it follows that each has four values in appearance only, two and two of each being identical.

Let the equations be $\left.\begin{array}{r} x + y = 6 \\ \text{and} \quad 2x - xy + 2y = 4 \end{array}\right\}$ To find the values of x and y.

Assuming $x = v + z$, and $y = v - z$, whence $x + y = 2v = 6$, or $v = 3$ from the first equation, and from the second equation we have $2v + 2z - v^2 + z^2 + 2v - 2z = 4$.

By substitution, $\quad 12 - 9 + z^2 = 4$

that is, $\quad z^2 = 1$

$\therefore \quad z = \pm 1$

Whence $x = 3 \pm 1 = 4$, or 2

and $y = 3 \mp 1 = 2$, or 4.

EQUATIONS.	ANSWERS.
1. $\left.\begin{aligned} x + y &= 2s \\ x - y &= 2z \\ x^2 + y^2 &= a \end{aligned}\right\}$	$\left\{\begin{aligned} x &= s + \sqrt{\frac{a - 2s^2}{2}}. \\ y &= s - \sqrt{\frac{a - 2s^2}{2}}. \end{aligned}\right.$
2. $\left.\begin{aligned} x + y &= 2s \\ x - y &= 2z \\ x^3 + y^3 &= b \end{aligned}\right\}$	$\left\{\begin{aligned} x &= s + \sqrt{\frac{b - 2s^3}{2}}. \\ y &= 0 - \sqrt{\frac{b - 2s^3}{2}}. \end{aligned}\right.$
3. $\left.\begin{aligned} x^2 + y^2 &= 58 \\ 3xy &= 63 \end{aligned}\right\}$	$\left\{\begin{aligned} x &= \pm 7, \text{ or } \pm 3. \\ y &= \pm 3, \text{ or } \pm 7. \end{aligned}\right.$
4. $\left.\begin{aligned} 2x^2 + 2y^2 &= 40 \\ x - 2xy + y &= -10 \end{aligned}\right\}$	$\left\{\begin{aligned} x &= \pm 4, \text{ or } \pm 2. \\ y &= \pm 2, \text{ or } \pm 4. \end{aligned}\right.$
5. $\left.\begin{aligned} x^2 + 2xy + y^2 &= 81 \\ x^2 - 2xy + y^2 &= +9 \end{aligned}\right\}$	$\left\{\begin{aligned} x &= \pm 6, \text{ or } \pm 3. \\ y &= \pm 3, \text{ or } \pm 6. \end{aligned}\right.$
6. $\left.\begin{aligned} x + y &= 5 \\ x^4 + y^4 &= 97 \end{aligned}\right\}$	$\left\{\begin{aligned} x &= 3,\ 2;\ \text{or } \frac{5 \pm \sqrt{-151}}{2} \\ y &= 2,\ 3;\ \text{or } \frac{5 \mp \sqrt{-151}}{2} \end{aligned}\right.$

Equations which do not come under any of these three cases may sometimes be solved by some of those artifices frequently made use of in analytical operations, and which can only be learned from experience.

Thus, let the equation be

$$\left.\begin{aligned} x^2 + y^2 &= 244 \\ \text{and } xy &= 120 \end{aligned}\right\} \text{To find the values of } x \text{ and } y.$$

Multiply the second equation by 2, $2xy = 240$

Add and subtract this from the first, and we have $\left\}\begin{aligned} x^2 + 2xy + y^2 &= 484 \\ x^2 - 2xy + y^2 &= 4 \end{aligned}\right.$

Take the root of both and we have . . $\left\}\begin{aligned} x + y &= \pm 22 \\ x - y &= \pm 2 \end{aligned}\right.$

Add and subtract and we have . . . $\left.\begin{array}{l} 2x = \pm 24, \text{ or } x = \pm 12. \\ 2y = \pm 20, \text{ or } y = \pm 10. \end{array}\right.$

Let the equations be $\left.\begin{array}{l} x^2y + xy = 120 \\ \text{and} \quad x^3y + y = 390 \end{array}\right\}$ To find the values of x and y.

From the first equation $\quad y = \dfrac{120}{x^2 + x}$

And from the second equation $y = \dfrac{390}{x^3 + 1}$.

$$\therefore \quad \frac{120}{x^2 + x} = \frac{390}{x^3 + 1}$$

Dividing by $\dfrac{30}{x + 1}$, $\quad \dfrac{4}{x} = \dfrac{13}{x^2 - x + 1}$

Clearing of fractions, $\quad 4x^2 - 4x + 4 = 13x$

Or $\quad 4x^2 - 17x = -4$

Divide by 4, $\quad x^2 - \dfrac{17}{4}x = -1$

Complete the square, $\quad x^2 - \dfrac{17}{4}x + \dfrac{289}{64} = \dfrac{289}{64} - \dfrac{64}{64} = \dfrac{225}{64}$

Extract the root, $\quad x - \dfrac{17}{8} = \pm \dfrac{15}{8}$

$$\therefore \quad x = \frac{17}{8} \pm \frac{15}{8} = 4, \text{ or } \tfrac{1}{4}$$

$$\text{and} \quad y = \frac{120}{x^2 + x} = \frac{120}{20}, \text{ or } \frac{120}{\frac{5}{16}} = 6, \text{ or } 384.$$

Let the equations be $\left.\begin{array}{l} y^2 - x^2 - (y + x) = 12 \\ \text{and} \quad (y - x)^2 \times (y + x) = 48 \end{array}\right\}$ To find the values of x and y.

Multiply the first equation by 4, $4\{y^2 - x^2 - (y + x)\} = 48$

Hence $4\{y^2 - x^2 - (y + x)\} = (y - x)^2 \times (y + x)$

Divide by $y + x$, $4(y - x - 1) = (y - x)^2$

Transpose, $\quad (y - x)^2 - 4(y - x) = -4$

Complete the square, $(y - x)^2 - 4(y - x) + 4 = 4 - 4 = 0$

Extract the root, $(y - x) - 2 = 0$

$$\therefore \quad y - x = 2$$

Substitute this value of $y - x$, in the second equation, and we have

$$4(y+x) = 48$$

$$\therefore \quad y + x = 12$$

And since $\quad y - x = 2$

We have by Addition and Subtraction, $\left.\begin{array}{l} 2y = 14, \text{ or } y = 7 \\ 2x = 10, \text{ or } x = 5 \end{array}\right.$

Let the equations be $\left.\begin{array}{rl} x^2 + y^2 &= 52 \\ \text{and} \quad 2xy - x - y &= 38 \end{array}\right\}$ To find the values of x and y.

Add the two equations and we have

$$x^2 + 2xy + y^2 - (x+y) = 90$$

$$\text{or} \quad (x+y)^2 - (x+y) = 90$$

Now let $v = x + y$, then $(x+y)^2 = v^2$

$$\text{and} \quad v^2 - v = 90$$

Complete the square, $v^2 - v + \frac{1}{4} = \frac{360}{4} + \frac{1}{4} = \frac{361}{4}$

Extract the root, $\quad v - \frac{1}{2} = \pm \frac{19}{2}$

$$\therefore \quad v = \frac{1}{2} \pm \frac{19}{2} = 10, \text{ or } -9.$$

that is, $\quad x + y = 10, \text{ or } -9$

$$\begin{array}{lrl}
\therefore & (x+y)^2 = & x^2 + 2xy + y^2 = 100, \text{ or } 81 \\
\text{Twice the first equation,} & = & 2x^2 \qquad + 2y^2 = 104, \,..\, 104 \\
\text{Subtract} & & x^2 - 2xy + y^2 = \quad 4, \,..\, 25 \\
\text{Extract the root,} & & x - y = \pm 2, \text{ or } \pm 5 \\
\text{but} & & x + y = +10, \,..\, -9
\end{array}$$

$$\text{Add and subtract} \quad \left\{\begin{array}{ll} 2x & = 12, \; 8, -14, \text{or} -4 \\ 2y & = 8, 12, -4, \,..\, -14 \end{array}\right.$$

$$\begin{array}{rll}
\therefore & x & = 6, \; 4, -7, \,..\, -2 \\
\text{and} & y & = 4, \; 6, -2, \,..\, -7.
\end{array}$$

EQUATIONS.	ANSWERS.
1. $\left.\begin{array}{l} x^2 + 2xy + 3x + y = 15 \\ x + 3y + y^2 = 6 \end{array}\right\}$	$\left\{\begin{array}{l} x = 2 \\ y = 1. \end{array}\right.$
2. $\left.\begin{array}{l} x^2 + 4xy + 4y^2 = 256 \\ 4y^2 - x^2 = 128 \end{array}\right\}$	$\left\{\begin{array}{l} x = 4 \\ y = 6. \end{array}\right.$
3. $\left.\begin{array}{l} x^2 + y^2 + x - y = 14 \\ (x-y) \times (x^2 + y^2) = 13 \end{array}\right\}$	$\left\{\begin{array}{l} x = 3 \\ y = 2. \end{array}\right.$
4. $\left.\begin{array}{l} x^2 - 4xy + y^2 = -12 \\ xy = 8 \end{array}\right\}$	$\left\{\begin{array}{l} x = 4, 2, -2, \text{ or } -4 \\ y = 2, 4, -4, \text{ or } -2. \end{array}\right.$

QUESTIONS PRODUCING QUADRATIC EQUATIONS, INVOLVING TWO UNKNOWN QUANTITIES.

1. There is a certain number consisting of two digits, the left hand digit is equal to twice the right hand one, and if 26 be subtracted from the number itself, the remainder is the square of the left hand digit. What is the number?

Let $x =$ the left hand digit, and $y =$ the other, then the number is $\quad 10x + y$

And by the question $\quad x = 2y$

Also $\quad 10x + y - 26 = x^2$.

Substituting the value of x in this last we have

$$20y + y - 26 = 4y^2$$

$$\therefore \quad 4y^2 - 21y = -26$$

$$\text{Or} \quad y^2 - \frac{21}{4}y = -\frac{13}{2}.$$

Complete the square, $y^2 - \frac{21}{4}y + \frac{441}{64} = \frac{441}{64} - \frac{416}{64} = \frac{25}{64}$.

Extract the root, $\quad y - \frac{21}{8} = \pm\frac{5}{8}$

$$\therefore \quad y = \frac{21}{8} \pm \frac{5}{8} = \frac{26}{8}, \text{ or } \frac{16}{8} = 3\tfrac{1}{4}, \text{ or } 2.$$

Here it is evident that if $3\frac{1}{4}$ be taken as the value of y, it will not fulfil the conditions of the question; therefore 2 is the proper value, and since $x = 2y$, $x = 4$, and the number is 42.

2. What two numbers are those whose sum multiplied by the greater makes 54, and whose difference multiplied by the less makes 9?

Let $x =$ the greater and $y =$ the less.

$$\left.\begin{array}{ll}\text{Then} & (x+y)x = x^2 + xy = 54 \\ \text{and} & (x-y)y = xy - y^2 = 9\end{array}\right\} \text{By the question.}$$

Assume $x = vy$, then $v^2y^2 + vy^2 = 54$, or $y^2 = \frac{54}{v^2+v}$

and $\quad vy^2 - y^2 = 9$, or $y^2 = \frac{9}{v-1}$

$$\text{Hence } \frac{54}{v^2+v} = \frac{9}{v-1}$$

Or $54v - 54 = 9v^2 + 9v$

Transposing, &c. $9v^2 - 45v = -54$

Divide by 9, $v^2 - 5v = -6$

Complete the square, $v^2 - 5v + \frac{25}{4} = \frac{25}{4} - \frac{24}{4} = \frac{1}{4}$

Extract the root, $v - \frac{5}{2} = \pm\frac{1}{2}$

$\therefore \quad v = \frac{5}{2} \pm \frac{1}{2} = 3$, or 2.

On trial we find that $v = 3$ will not answer the conditions of the question; therefore taking $v = 2$ we have

$$y^2 = \frac{9}{v-1} = 9$$

$\therefore \quad y = \pm 3$

and $\quad x = vy = 2 \times 3 = 6$.

3. What two numbers are those of which the sum is 8, and the sum of their fifth powers 3368?

Let the greater $x = v+z$, and the less $y = v - z$, then by the question $2v = 8$, or $v = 4$; hence

$$x^5 = (v+z)^5 = v^5 + 5v^4z + 10v^3z^2 + 10v^2z^3 + 5vz^4 + z^5$$
$$y^5 = (v-z)^5 = v^5 - 5v^4z + 10v^3z^2 - 10v^2z^3 + 5vz^4 - z^5$$

Then, by the question, $2v^5 \quad * \quad +20v^3z^2 \quad * \quad +10vz^4 = 3368$

Or since $v = 4$, $2048 + 1280z^2 + 40z^4 = 3368$

Divide by 8, $256 + 160z^2 + 5z^4 = 421$

Transpose, $5z^4 + 160z^2 = 421 - 256 = 165$

Divide by 5, $z^4 + 32z^2 = 33$

Complete the sq. $z^4 + 32z + 256 = 33 + 256 = 289$

Extract the root, $z^2 + 16 = \pm 17$

$\therefore \quad z^2 = 17 - 16 = 1$

and $\quad z = 1$

Whence $x = v + z = 4 + 1 = 5$

and $\quad y = v - z = 4 - 1 = 3$.

4. A rectangular bowling-green is of such dimensions, that had it been 5 feet longer and 6 broader, it would have contained 590 square feet more, and had it been 6 feet longer and 5 broader it would have contained only 570 square feet more. Required its length and breadth.

Ans. 60 feet long and 40 feet broad.

5. What number is that which being divided by the product of its two digits the quotient is 3, and if 18 be added to it the digits will be reversed? Ans. 24.

6. What number is that which being divided by the sum of its two digits the quotient is 7, and when divided by the difference of its digits the quotient is one-half of the number?

Ans. 42.

7. There are three numbers whose sum is 46, the difference of their differences is 4, and the sum of their squares is 836. What are the numbers? Ans. 8, 14, and 24.

8. What three numbers are those whose sum is 29, whose continued product is 576, and the difference of their differences is 2? Ans. 4, 9, and 16.

9. The sum of four numbers in arithmetical progression is 20, and the sum of their squares 120. Required the numbers.

Ans. 2, 4, 6, and 8.

10. The sum of three numbers in geometrical progression is 28, and the sum of their squares 360. Required the numbers. Ans. 4, 8, and 16.

11. There are three numbers in geometrical progression such that their sum is 14, and the sum of their squares 84. What are the numbers? Ans. 2, 4, and 8.

12. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; but if the periphery of each wheel be increased 1 yard, it will make only 4 revolutions more than the hind-wheel in going over the same space. Required the circumference of each.

Ans. fore-wheel 4 yds. hind-wheel 5 yds.

13. What two numbers are those whose sum multiplied by the greater is 700, and whose difference multiplied by the less is 75? Ans. 15 and 20.

14. Required two numbers such that their product shall be equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes.

Ans. $\frac{1}{2}\sqrt{5}$ and $\frac{1}{4}(5+\sqrt{5})$.

15. A and B engage to reap a field for £4, 10s.; and as A alone would reap it in 9 days, they promise to complete it in 5 days. They found, however, that they could not do it in this time, and were obliged to call in C, an inferior workman, to assist them for the two last days, in consequence of

which B receives 3s. 9d. less than he otherwise would have done. In what time could B and C alone reap the field?

Ans. B alone in 15 days, and C alone in 18 days.

16. A person bought two cubical stacks of hay for £41, each of which cost as many shillings per solid yard as there were yards in a side of the other, and the greater stood on more ground than the less by 9 square yards. What was the price of each? Ans. the greater £25, and the less £16.

17. A and B put out different sums to interest, amounting together to £200. B's rate of interest was one per cent. more than A's, and at the end of 5 years B's accumulated simple interest wanted only £4 to be double of A's. At the end of 10 years, A's amount was to that of B as 5 : 8. Required the sum put out by each and the rates per cent.

Ans. A's principal 80, and B's £120; A's rate per cent. 5, and B's 6.

18. Two detachments of foot being ordered to a station 39 miles distant, began their march at the same time, and the one by marching a quarter of a mile per hour quicker than the other, arrived there an hour sooner. Required their rates of marching. Ans. $3\frac{1}{4}$ and 3 miles an hour.

19. Required four numbers in geometrical progression such that their sum shall be 40, and the sum of their squares 820. Ans. 1, 3, 9, and 27.

20. Required two numbers such that the square of the first added to their product shall be 65, and the square of the second *minus* their product shall be 24. Ans. 5 and 8.

21. Required two numbers whose sum is 5, and the sum of their fifth powers 275. Ans. 2 and 3.

22. Required two numbers whose difference is 2, and the difference of their fourth powers 1040. Ans. 4 and 6.

23. A gentleman gave £12 to be distributed among some poor people upon his estate; but before the distribution took place four additional claimants appeared, by which means the former received 2s. a-piece less than they would have otherwise done. What was the number at first? Ans. 20.

24. Find four numbers in arithmetical progression such that their common difference shall be 3, and their continued product 1944. Ans. 3, 6, 9, and 12.

25. A grocer sold 80 lbs. of mace and 100 lbs. of cloves for £65, and found that he had sold 60 lbs. more of cloves for £20, than of mace for £10. What was the price of a pound of each? Ans. Mace 10s. per lb., and cloves 5s. per lb.

26. Find four numbers in arithmetical progression, the product of whose extremes is 40, and the product of whose means is 48. Ans. 4, 6, 8, and 10.

27. Find five numbers in arithmetical progression whose sum is 30 and whose product is 3840.

Ans. 2, 4, 6, 8, and 10.

28. A company at a tavern had £10 to pay for their reckoning; but four of them having no money, the remainder had to pay 13s. 4d. a-piece more than their just share. What was the number of the company? Ans. 10.

29. There are two numbers whose sum, product, and difference of their squares are all equal to one another. What are those numbers? Ans. $\frac{1}{2}+\sqrt{\frac{5}{4}}$, and $\frac{3}{2}+\sqrt{\frac{5}{4}}$.

30. When the price of brandy was three times the price of British spirits, a merchant made two mixtures of brandy and British spirit, and the prices per gallon were in the ratio of 9 to 10. He afterwards mixed twice as much brandy with the same quantity of British spirit in each case, and the relative price was the same as before. Required the ratio of the quantities mixed.

Ans. the 1st mixture was in the ratio of 3 to 1, and 2 to 1; the 2d mixture was in the ratio of 3 to 2, and of equality.

OF CUBIC AND HIGHER EQUATIONS.

In considering equations, it is supposed that all the terms are brought to one side, or the equation made $=0$, and the terms arranged according to the powers of the unknown quantity, the highest being placed on the left hand, and the others in their order; and the absolute term, or that into which the unknown quantity does not enter, being placed last, or on the right hand.

If any number of simple equations be multiplied together, an equation will be formed, of which the highest index is the number of simple equations; thus $x-a=0 \times x+b=0 \times x-c=0$ produce $x^3+x^2(b-a-c)+x \times (ac-ab-bc)+abc$. Hence every equation is understood to be composed of as many simple equations as there are units in the highest exponent.

In a complete equation, all the powers of the unknown quantity are found, and the number of terms is one greater than the highest exponent.

When the coefficient of the highest term is 1, the coefficient of the second term is the sum of all the roots or values of the unknown quantity; the coefficient of the third term is the sum of all the products of the roots taken two by two;

the coefficient of the fourth term is the sum of all the products taken three and three, &c.; and the absolute term is the continued product of all the roots.

A quadratic equation, as has been already shown, contains *two* values, a cubic equation contains *three* values, a biquadratic *four* values, and an equation of n dimensions contains n values of the unknown quantity and no more.

When an equation contains both positive and negative roots, it will have as many positive roots as it has changes of the signs from $+$ to $-$, or from $-$ to $+$, and as many negative roots as it has continuations of the same sign $+$ to $+$, or $-$ to $-$, among its coefficients; thus, in the equation $x^4 + 9x^3 + 9x^2 - 41x - 42 = 0$, there is but one change of the signs from $+$ to $-$, and three continuations of the same sign, namely two from $+$ to $+$, and one from $-$ to $-$; and the equation has therefore one positive and three negative roots, namely, $2, -1, -3$, and -7.

If one or more roots of an equation be known, it may be divided by the equation which contains these known roots, and the quotient will be an equation which contains the rest of the roots of the original equation; thus if the cubic equation $x^3 + 9x^2 + 23x - 15 = 0$, one of whose roots is $+1$, be divided by $x - 1 = 0$, the quotient is $x^2 - 8x + 15 = 0$, a quadratic equation whose roots are 4 ± 1, or 3 and 5, the three roots of the original equation are therefore 1, 3, and 5.

TRANSFORMATION OF EQUATIONS.

I. To change the signs of the roots.

All the positive roots of an equation may be changed into negative, and all the negative into positive ones, by changing the signs of the 2d, 4th, &c. terms of the equation; for the coefficients of these terms are combinations of an odd number of the roots.

NOTE. Impossible roots are here supposed to be either positive or negative.

Thus the roots of the equation $x^4 - 2x^3 - 13x^2 + 38x - 24 = 0$ are $+1, +2, +3$, and -4, and by changing the signs of the 2d and 4th terms we have $x^4 + 2x^3 - 13x^2 - 38x - 24 = 0$, whose roots are $-1, -2, -3$, and $+4$. The roots of $x^5 - 5x^4 - 70x^3 + 230x^2 + 789x - 945 = 0$, are $+1, -3 + 5, -7$, and $+9$, and by changing the signs of the 2d, 4th, and 6th terms, we have $x^5 + 5x^4 - 70x^3 - 230x^2 + 789x + 945 = 0$, whose roots are $-1, +3, -5, +7$, and -9.

II. To take away the second term.

Divide the coefficient of the second term by the highest

exponent of the unknown quantity, and annex the quotient with its sign changed to another unknown quantity, and substitute the amount and its powers for the original unknown quantity, and its powers, in the given equation, and a new equation will arise wanting the second term.

Transform the equation $x^3 - 9x^2 + 26x - 34 = 0$ into another wanting its second term.

Here $x = y + \frac{9}{3} = y + 3$

$$\text{And}\left\{\begin{array}{rcl} x^3 &=& y^3 + 9y^2 + 27y + 27 \\ -\ 9x^2 &=& \quad - 9y^2 - 54y - 81 \\ 26x &=& \qquad\qquad + 26y + 78 \\ -\ 34 &=& \qquad\qquad\qquad\quad - 34 \end{array}\right.$$

$$y^3 \quad * \quad - \quad y - 10 = 0.$$

where y^2, the second term of the equation, is wanting.

Transpose the equation $x^4 + 4x^3 - 34x^2 - 76x + 105 = 0$, into another wanting its second term.

Here $x = y - \frac{4}{4} = y - 1$

$$\text{And}\left\{\begin{array}{rcl} x^4 &=& y^4 - 4y^3 + 6y^2 - 4y + 1 \\ 4x^3 &=& \quad + 4y^3 - 12y^2 + 12y - 4 \\ -\ 34x^2 &=& \qquad\quad - 34y^2 + 68y - 34 \\ -\ 76x &=& \qquad\qquad\quad - 76y + 76 \\ +\ 105 &=& \qquad\qquad\qquad\quad + 105 \end{array}\right.$$

$$y^4 \quad * \quad - 40y^2 \quad * \quad + 144 = 0.$$

Transform the equation $x^3 + 9ax^2 - b = 0$ into another wanting its second term.

Here $x = y - \frac{9a}{3} = y - 3a$

$$\text{And}\left\{\begin{array}{rcl} x^3 &=& y^3 - 9ay^2 + 27a^2y - 27a^3 \\ 9ax^2 &=& \quad + 9ay^2 - 54a^2y + 81a^3 \\ -\ b &=& \qquad\qquad\qquad\qquad - b \end{array}\right.$$

$$y^3 \quad * \quad - 27a^2y + 54a^3 - b = 0.$$

1. Transform the equation $x^3 - 3x^2 + 3x + 12 = 0$, into another wanting its second term.

Ans. $y^3 + 13 = 0$.

2. Transform the equation $x^3 - 15x^2 + 81x - 243 = 0$, into another wanting its second term.

Ans. $y^3 + 6y - 88 = 0$.

3. Transform the equation $x^3 + 6x^2 + 4x - 27 = 0$, into another wanting its second term.

Ans. $y^3 - 8y - 19 = 0$.

4. Transform the equation $x^5 - 5x^4 - 70x^3 + 230x^2 + 789x - 945 = 0$, into another wanting its second term.

Ans. $y^5 - 80y^3 + 1024y = 0$.

5. Transform the equation $x^5 + 3x^4 - 23x^3 - 27x^2 + 166x - 120 = 0$, into another wanting its second term.

$$\text{Ans. } y^5 - \frac{133y^3}{5} + \frac{468y^2}{25} + \frac{21452y}{125} - \frac{700128}{3125} = 0.$$

6. Transform the equation $x^4 - 2x^3 - 13x^2 + 38x - 24 = 0$, into another wanting its second term.

Ans. $y^4 - 14{\cdot}75y^2 + 23{\cdot}75y - 8{\cdot}4375 = 0$.

III. To clear an equation of fractions, and to make the coefficient of the highest term unity.

Any equation may be cleared of fractions by multiplying all its terms by the least common multiple of the denominators of the fractions, as already shown in simple equations, when, if the resulting equation be of the form $ax^4 - bx^3 + cx^2 - dx + e = 0$, the coefficient of its highest term must be made unity, before proceeding to its solution; to accomplish this,

Assume $x = \frac{y}{a}$ and substitute $\frac{y}{a}$ and its power, instead of x and its powers in the given equation, and we obtain $\frac{y^4}{a^3} - \frac{by}{a^3} + \frac{cy^2}{a^2} - \frac{dy}{a} + e = 0$; multiply this by a^3 and we have $y^4 - by^3 + acy^2 - a^2dy + a^3c = 0$, an equation in which the coefficient of y is unity, and whose roots are a times the roots of the original equation.

Transform the equation $3x^3 - 13x^2 + 14x + 16 = 0$, into another in which the coefficient of the highest term is unity.

Here $a = 3$ and $x = \frac{y}{3}$, and substituting this value of x and its powers in the given equation, we have $\frac{y^3}{3^2} - \frac{13y}{3^2} + \frac{14y}{3} + 16 = 0$, and multiplying this by 3^2 or 9, we have $y^3 - 13y^2 + 42y + 144 = 0$, an equation in which the coefficient of y is unity, and its roots 3 times the roots of the original equation.

Transform the following equations into others whose coefficients shall be integral, and that of the highest term unity.

EQUATIONS.	ANSWERS.
1. $x^3 - \frac{1}{2}x^2 + \frac{2}{3}x + 1 = 0.$	$y^3 - 3y^2 + 24y + 216 = 0.$
2. $x^3 - 3x^2 + \frac{11}{4}x - \frac{3}{4} = 0.$	$y^3 - 6y^2 + 11y - 6 = 0.$
3. $5x^3 + 7x^2 - 24x - 35 = 0.$	$y^3 + 7y^2 - 120y - 785 = 0.$

4. $7x^4 - 3x^3 - 27x^2 + 15x - 44 = 0.$

Ans. $y^4 - 3y^3 - 189y^2 + 735y - 15092 = 0.$

5. $x^3 - \frac{3x^2}{2} + \frac{2}{3}x - \frac{3}{4} = 0.$

Ans. $y^3 - 18y^2 + 96y - 1296 = 0.$

6. $vx^4 - px^3 + qx^2 - rx + s = 0.$

Ans. $y^4 - py^3 + vqy^2 - v^2ry + v^3s = 0.$

SOLUTION OF EQUATIONS OF ALL DEGREES WHICH HAVE RATIONAL ROOTS.

Find all the divisors of the absolute term, and substitute them with the signs $+$ and $-$ in the given equation, instead of the unknown quantity, and all those which make the equation equal to 0 are roots of the equation; thus, in the equation $x^3 - 3x^2 - 13x + 15 = 0$, the divisors of 15, the absolute term, are 1, 3, and 5; suppose then $x = +1$, the equation will be $1 - 3 - 13 + 15 = 0$, so that $+1$ is a value of x. Again, suppose $x = -3$ and the equation will be $-27 - 27 + 39 + 15 = 0$; hence -3 is also a value of x. Lastly, suppose $x = +5$, then the equation is $125 - 75 - 65 + 15 = 0$; therefore $+5$ is also a value of x.

To find the values of x in the equation $x^4 + 8x^3 - 49x^2 - 8x + 48 = 0$.

To find all the simple divisors of the absolute term 48, divide it by 2 as often as possible, then divide it by 3 as often as possible, and in like manner by 5, 7, &c. Here all the simple divisors are, 1, 2, 2, 2, 2, 3, the composite divisors are found by multiplying any two or more of the simple ones, and are 4, 6, 8, 12, 16, 24, 48. The whole of these divisors must now be tried with the signs $+$ and $-$.

When the divisors of the absolute terms are very numerous it becomes then a work of great labour to try them all. Those to be tried may however be reduced to a small number by the following rule.

Substitute for the unknown quantity successively three or more terms of the progression 2, 1, 0, — 1, — 2, and find all the divisors of the results, then select from them all the progressions, in a vertical direction, of which the common difference is unity; and the roots of the equation will be such terms of these progressions as arise from the substitution of 0 for the unknown quantity, and they are positive if the progressions increase from the top to the bottom, or negative if they decrease.

If the number of progressions be still too numerous, they may be farther limited by substituting more terms of the progression 2, 1, 0, — 1, — 2, &c.

Required the roots of the equation $x^4 - 2x^3 - 25x^2 + 26x + 120 = 0$.

Suppose	Results.	Divisors.	Progressions.
$x = +2$	$+72$	1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, &c.	1, 4, 3, 6
$x = +1$	$+120$	1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, &c.	2, 3, 4, 5
$x = 0$	$+120$	1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, &c.	3, 2, 5, 4
$x = -1$	$+72$	1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, &c.	4, 1, 6, 3

Here the numbers standing in the same line with 0 in suppositions are 3, 2, 5, and 4; these are therefore to be taken and substituted in the given equation, when we find that — 2, + 3, — 4, and + 5 are its four roots.

Required the roots in the equation $x^4 + 3x^3 + 2x^2 - 10x + 28 = 0$.

Suppose	Results.	Divisors.	Progression.
$x = +2$	$+56$	1, 2, 4, 7, 8, 14, 28, 56	4
$x = +1$	$+24$	1, 2, 3, 4, 6, 8, 12, 24	3
$x = 0$	$+28$	1, 2, 4, 7, 14, 28	2
$x = -1$	$+38$	1, 2, 19, 38	1

Here the number 2 standing on the same line with 0 is to be tried with a negative sign, as the progression decreases, but upon trial it is found not to succeed; the equation has therefore no rational root.*

* In all cases where progressions are found among the divisors, which do not succeed as roots of the equation, it will be found that the first progression 2, 1, 0, — 1, &c. has not been carried far enough; thus, in the present equation, had another term been taken in the first, there would have been no progression among the divisors, and, consequently, we should at once have seen that there was no rational root to the equation.

Find the values of x in the following equations.

EQUATIONS.	ANSWERS.
1. $x^3 - x^2 - 10x + 6 = 0.$	$x = -3$, the only rational value.
2. $x^3 - 3x^2 - 46x - 72 = 0.$	$x = -2, -4, +9.$
3. $x^3 - 2x^2 - 33x + 90 = 0.$	$x = +3, +5, -6.$
4. $x^4 + 2x^3 - 41x^2 - 42x + 360 = 0.$	$x = +3, -4, +5, -6.$
5. $x^4 - 6x^3 - 8x^2 + 51x - 308 = 0.$	$x = -4, +7.$
6. $x^5 - 11x^4 + 5x^3 + 131x^2 - 102x - 216 = 0.$	$x = -1, +2, -3, +4, +9.$
7. $x^5 + x^4 - 14x^3 - 6x^2 + 20x + 48 = 0.$	$x = +2, +3$, and -4, the only rational values.
8. $x^4 + x^3 - 29x^2 - 9x + 180 = 0.$	$x = +3, +4, -3, -5.$

GENERAL SOLUTION OF CUBIC EQUATIONS.

I. *Of Pure Cubic Equations.*

A cubic equation is said to be pure when the cube of the unknown quantity is equal to a known quantity, in which case the equation is

$$x^3 - a = 0, \ - x^3 - \frac{a}{b} = 0, \text{ or } x^3 - 64 = 0.$$

It is manifest, that in order to find a value of x, we have only to extract the cube root of the equation. Thus $x^3 - 343$, or $x^3 = 343$, gives $x = 7$, one of the roots of the equation.

It has, however, been already shown that an equation of the third degree has three roots; in order therefore to find the other two roots of the equation, divide it by $x - 7 = 0$ and we have

$$\begin{array}{l}
x - 7)x^3 - 343(x^2 + 7x + 49 \\
\quad\quad\quad x^3 - 7x^2 \\
\hline
\quad\quad\quad\quad + 7x^2 - 343 \\
\quad\quad\quad\quad + 7x^2 - 49x \\
\hline
\quad\quad\quad\quad\quad\quad + 49x - 343 \\
\quad\quad\quad\quad\quad\quad + 49x - 343 \\
\hline
\end{array}$$

but $x^2 + 7x + 49$ is a quadratic the two values of which are

$$x = -\frac{7}{2} + \sqrt{-\frac{147}{4}}, \text{ and } x = -\frac{7}{2} - \sqrt{-\frac{147}{4}}.$$

Let the equation be $x^3 - 8 = 0$, one of the roots here is 2;

Hence $\quad x - 2)x^3 - 8(x^2 + 2x + 4$

$$
\begin{array}{l}
x^3 - 2x^2 \\
\hline
\qquad 2x^2 - 8 \\
\qquad 2x^2 - 4x \\
\hline
\qquad\qquad 4x - 8 \\
\qquad\qquad 4x - 8 \\
\hline
\end{array}
$$

and the two values of the quadratic $x^2 + 2x + 4$ are $x = -1 + \sqrt{-3}$ and $-1 - \sqrt{-3}$.

Let $x^3 - 27 = 0$.

Ans. $x = 3, -\frac{3}{2} + \sqrt{-\frac{27}{4}}, -\frac{3}{2} - \sqrt{-\frac{27}{4}}$.

Let $x^3 + 125 = 0$.

Ans. $x = -5, \frac{5}{2} + \sqrt{-\frac{75}{4}}, \frac{5}{2} - \sqrt{\frac{75}{-4}}$.

Let $x^3 - 512 = 0$.

Ans. $x = 8, -4 + \sqrt{-48}, -4 - \sqrt{-48}$.

Let $x^3 + 216 = 0$.

Ans. $x = -6, 3 + \sqrt{-27}, 3 - \sqrt{-27}$.

II. *Solution of Complete Cubic Equations.*

If the equation has all its terms, the second must be taken away, and the coefficient of the highest term made unity; it will then be of the form, $x^3 + 3qx + 2r = 0$, where the sign $+$ denotes that the term is to be added with its proper sign.

Assume $x = v + z$ and $q = vz$, substituting these in the equation $x^3 + 3qx + 2r = 0$, and reducing we obtain $v^6 - 2rv^2 - q^3 = 0$, a quadratic, which being resolved, gives $v = \sqrt[3]{(-r + \sqrt{r^2 + q^3})}$, and $z = \sqrt[3]{-z - \sqrt{r^2 + q^3}}$; therefore $x = \sqrt[3]{\{(-z + \sqrt{r^2 + q^3}) + \sqrt[3]{(-z - \sqrt{r^2 + q^3})}\}}$, or because $z = \frac{q}{v}$, then $x = \sqrt[3]{(-r + \sqrt{r^2 + q^3} - \frac{q}{\sqrt[3]{-r - \sqrt{r^2 + q^3}}})}$.

If q is negative, and r^2 less than q^3, then the quantity under the radical sign $\sqrt{\ }$ becomes negative, and its root im-

possible, though in this case x may have three real roots; so that this formula fails, except only when x has two imaginary roots.*

Find the value of y in the equation $y^3+3y^2+9y-13=0$.

Assume $x-1=y$, and substitute this value of y and its powers in the given equation, and we obtain $x^3+6x-20=0$, an equation wanting its second term, where $+q=+2$ and $+r=-10$.

Therefore $x=\sqrt[3]{\{+10+\sqrt{100+8}\}}-\dfrac{2}{\sqrt[3]{\{+10+\sqrt{(100+8)}\}}}=$

$\sqrt[3]{(10+10\cdot3923)}-\dfrac{2}{\sqrt[3]{(10+10\cdot3923)}}=2\cdot73205-\cdot73205=2$;

hence $y=x-1=2-1=1$.

Dividing the given equation by $y-1$ we obtain $y^2+4y+13=0$, a quadratic the two roots of which are $y=-2\pm\sqrt{-9}$; the three roots of the equation are therefore 1, $-2+\sqrt{-9}$, and $-2-\sqrt{-9}$, the two last of which are imaginary.

Find the value of x in the equation $x^3-6x-9=0$.

Here $+q=-2$ and $+r=-\dfrac{9}{2}$; therefore

$$x=\sqrt[3]{\left(\frac{9}{2}+\sqrt{\frac{81}{4}-8}\right)}-\frac{-2}{\sqrt[3]{\left(\frac{9}{2}+\sqrt{\frac{81}{4}-8}\right)}}=\sqrt[3]{\left(\frac{9}{2}+\frac{7}{2}\right)}$$

$$-\frac{-2}{\sqrt[3]{\left(\frac{9}{2}+\frac{7}{2}\right)}}=\sqrt[3]{8}-\frac{-2}{\sqrt[3]{8}}=2-\frac{-2}{2}=2+1=3.$$

Dividing the given equation by $x-3$, we obtain x^2+3x+3, a quadratic whose roots are $x=-\dfrac{3}{2}\pm\sqrt{-\dfrac{3}{4}}$; the

---

* The formula, which is usually called Cardan's Rule, is, if $x^3+ax=b$,

$$x=\sqrt[3]{\left\{\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}+\frac{a_3}{27}\right)}\right\}}+\sqrt[3]{\left\{\frac{b}{2}-\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}}\text{ or,}$$

$$x=\sqrt[3]{\left\{\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}+\frac{a^4}{27}\right)}\right\}}-\frac{\frac{1}{3}a}{\sqrt[3]{\left\{\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}}}.$$

three roots of the equation are therefore 3, $-\frac{3}{2}+\sqrt{-\frac{3}{4}}$, and $-\frac{3}{2}-\sqrt{-\frac{3}{4}}$.

Find the value of x in the following equations.

EQUATIONS.	ANSWERS.
1. $x^3-6x+9=0$.	$x=-3$.
2. $x^3-15x-4=0$.	$x=4$.
3. $x^3+24x-587914=0$.	$x=83{\cdot}6773$.
4. $x^3-6x^2+18x-22=0$.	$x=2{\cdot}3275$.
5. $x^3+2x^2-23x-70=0$.	$x=5{\cdot}1349$.
6. $x^3-9x^2+26x-24=0$.	$x=2$.
7. $x^6-3x^4-2x-8=0$.	$x=2$.
8. $x^3-5x-33074556=0$.	$x=321$.
9. $x^3-5x-1=0$.	$x=2{\cdot}33005874$.
10. $x^3-12x+15=0$.	$x=2{\cdot}396475$.
11. $x^3+7x-2=0$.	$x=0{\cdot}28249374$.
12. $x^3-125x-412500=0$.	$x=75$.
13. $x^3-9x+28=0$.	$x=-4$.
14. $x^3-6x+4=0$.	$x=2$.
15. $x^3-3x-4=0$.	$x=2{\cdot}2$.
16. $x^3+2x-12=0$.	$x=2$.
17. $x^3-6x^2-7x+60=0$.	$x=4$.
18. $x^3-5x^2+2x+12=0$.	$x=-3$.
19. $x^3-17x^2+54x-350=0$.	$x=14{\cdot}954068$.
20. $x^3-15x^2+71x-297=0$.	$x=11$.
21. $8x^3+24x-32=0$.	$x=1$.
22. $x^3-5x^2+\frac{17}{4}x-\frac{1}{4}=0$.	$x=1$.

SOLUTION OF CUBIC EQUATIONS BY A TABLE OF NATURAL SINES.

It has been formerly stated, that the formula of Cardan fails, except when x has two imaginary roots; but the value of x may in all cases be determined by the use of a table of natural sines; thus, in the equation

$$x^3+3qx+2r=0.$$

Find the square root of q, and divide the absolute term $2r$ by twice the cube of this root, that is by $2q^{\frac{3}{2}}$, and in a table of natural sines find the quotient and take the arc of which it is the cosine; then take the natural cosine of $\frac{1}{3}$ of this arc and multiply it by twice the square root of q, that is by $2q^{\frac{1}{2}}$, and it will give one of the values of x.

To find the other two values of x, add 120° and 240° to $\frac{1}{3}$ of the arc formerly found, and multiply the natural cosines of the sums by twice the square root of q.

Let the equation be $x^3 - 19x - 30 = 0$ to find the value of x.

Here $q = \frac{19}{3}$, $2r = 30$ and $\frac{r}{2q^{\frac{3}{2}}} = \frac{r}{2q^{\frac{3}{2}}} = 0{\cdot}941115$ the cosine of 19° 45′ 37·2″, and $\frac{1}{3}$ of this is 6° 35′ 12·4″, to which add 120°, and 240°, and we have 126° 35′ 12·4″, and 246° 35′ 12·4″ for the other two roots; the three cosines are therefore $+ {\cdot}9934$, $- {\cdot}5960$, and $- {\cdot}39736$, and these multiplied by $2\sqrt{q} = 5{\cdot}033224$ produce the three roots or values of x, namely, $x = +5$, -3, and -2.

Find the value of x in the following equations.

EQUATIONS.	ANSWERS.
1. $x^3 - 91x + 330 = 0$.	$x = +5, +6$, and -11.
2. $x^3 - 12x + 8 = 0$.	$x = \begin{cases} 3{\cdot}0641776 \\ 3{\cdot}7587704 \\ {\cdot}6945928. \end{cases}$
3. $x^3 - 2x + 2 = 0$.	$x = \begin{cases} -1{\cdot}7693 \\ \\ \end{cases}$
4. $x^3 - 9x + 9 = 0$.	$x = \begin{cases} 3{\cdot}411474 \\ -2{\cdot}226682 \\ -1{\cdot}184792. \end{cases}$
5. $x^3 - 3x - 1 = 0$.	$x = \begin{cases} 1{\cdot}879385 \\ -1{\cdot}532089 \\ -0.347296. \end{cases}$
6. $x^3 - 6x^2 + 18x - 22 = 0$.	$x = \begin{cases} 2{\cdot}3274 \\ \\ \end{cases}$
7. $x^3 - 12x + 10 = 0$.	$x = \begin{cases} {\cdot}89260 \\ 2{\cdot}93048 \\ -2{\cdot}82305. \end{cases}$

SOLUTION OF BIQUADRATIC EQUATIONS.

The general form of biquadratic equations, or those of the ırth degree, is

$$x^4 + ax^3 + bx^2 + cx + d = 0.$$

But before proceeding to the resolution of the general ıation, we shall show the method of resolving pure biqua- ıtic equations of which the form is

$$x^4 - a = 0.$$

Since x^4 is the square of x^2 we have $x^2 = \sqrt{a}$ and $x =$ ɩ.

Let the equation be $x^4 = 625$ to find the value of x. We re first $x^2 = \sqrt{625} = 25$ and $x = \sqrt{25} = 5$, which is : of the roots of the equation, but we have already stated ıt an equation of the fourth degree has four roots.

Resuming the equation $x^4 = 625$ we have not only $x^2 =$ but also $x^2 = -25$, whence from the first of this value have $x = 5$ and $x = -5$, and from the second $x = \sqrt{}$ 25 and $x = -\sqrt{-25}$, which are equal to $5\sqrt{-1}$ and $5\sqrt{-1}$, and these are the four roots or values of x in given equation.

Let the equation be $x^4 = 4096$.

Here $x^2 = \pm 64$

$\therefore \quad x = \pm 8$, or $\pm 8\sqrt{-1}$

at is, the four roots are $8, -8, 8\sqrt{-1}$, and $-8\sqrt{-1}$.

Biquadratic equations wanting the *second* and fourth ms, or such as are of the form $x^4 + bx^2 + d = 0$, may be ved by the rules for quadratic equations; thus, putting x^2 y we have

$$y^2 + by + d = 0$$

$$\text{Or } y^2 + by + \frac{b^2}{4} = \frac{b^2}{4} - d$$

$$\therefore \quad y + \frac{b}{2} = \pm\sqrt{\frac{b^2}{4} - d}$$

$$\text{Or } y = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - d} = \frac{-b \pm \sqrt{b^2 - 4d}}{2}$$

d since $x^2 = y$, $x = \pm\sqrt{\dfrac{-b \pm \sqrt{b^2 - 4d}}{2}}$, where the ble signs $\pm$ point out the four roots of the equation.

SOLUTION OF COMPLETE BIQUADRATICS.

From complete biquadratics we must first take away the second term, and the equation will then be of the form

$$x^4 + ax^2 + cx + d = 0.$$

Suppose this equation equal to $(x^2 + nx + r) \times (x^2 - nx + p)$; performing the multiplication we have $x^4 - (p + r - n^2)\, x^2 + (pn - rn)\, x + pr$ and comparing the coefficients of this with those of the former we observe that $p + r - n^2 = a$, or $p + r = a + n^2$; $pn - rn = c$, or $p - r = \frac{c}{n}$; hence $p = \frac{1}{2}\left(a + n^2 + \frac{c}{n}\right)$, and $r = \frac{1}{2}\left(a + n^2 - \frac{c}{n}\right)$; therefore $pr = d = \frac{1}{4}\left\{(a + n^2)^2 - \frac{c^2}{n^2}\right\}$, in which putting $v = n^2$ and reducing, becomes $v^3 + 2av^2 + (a^2 - 4d)\, v - c^2 = 0$, a cubic equation, from which v and consequently p and r may be found and substituted in the two quadratics $x^2 - nx + r$ and $x - nx + p$, which when resolved give the four roots of the given equation.

Find the four roots of the equation $x^4 - 25x^2 + 60x - 36 = 0$.

Here $p = \frac{1}{2}\left(n^2 + \frac{c}{n} - 25\right)$; $r = \frac{1}{2}\left(n^2 - \frac{c}{n} - 25\right)$, and $rp = d = \frac{1}{4}\left\{(n^2 - 25)^2 - \frac{60^2}{n^2}\right\} = -36$, putting $v = n^2$ and reducing, we have $v^3 - 50v^2 + 769v - 3600 = 0$, a cubic equation, of which the roots are rational, viz. 9, 16, and 25. Taking $v = 25$, then $n = 5$, $r = -6$, and $p = +6$, hence the two quadratics are $x^2 + 5x - 6 = 0$, and $x^2 - 5x + 6 = 0$, from the former we obtain $x = 1$, or -6, and from the latter $x = 2$ or 3.

It is a remarkable fact, that the other two values of v give the same values of x, but not in similar pairs. Thus, if $v = 9$, then $n = 3$, $r = -18$, and $p = 2$, hence the two quadratics are $x^2 + 3x - 18 = 0$, or $x = 3$, or -6, and $x^2 - 3x + 2 = 0$, or $x = 1$ or 2. Again, if $v = 16$, then $n = 4$, $r = -12$ and $p = 3$, hence the two quadratics are $x^2 + 4x - 12 = 0$, or $x = +2$, or -6, and $x^2 - 4x + 3 = 0$, or $x = 3$ or 1. It is therefore necessary to find only one of the roots of the cubic equation to obtain all the four values of x in the biquadratic.

l the four roots of the equation $x^4 - 4x^3 - 8x +$ 0.

stituting $y+1$ for x we get rid of the second term and $-6y^2 - 16y + 21 = 0$; here $r + p - n^2 = -6$, $r = n^2 - 6$, also $n(p - r) = -16$, or $p - r =$

hence $p = \frac{1}{2}(n^2 - 6 - \frac{16}{n}$, $r = \frac{1}{2}(n^2 - 6 + \frac{16}{n})$,

$= + 21 = \frac{1}{4}\left\{(n^2 - 6)^2 - \left(\frac{16}{n}\right)^2\right\}$, therefore re-

we have $n^6 - 12n^4 - 48n^2 - 256 = 0$, or sub-g $v = n^2$, $v^3 - 12v^2 - 48v - 256 = 0$, in which n we find $v = 16$, and consequently $n = 4$, whence 3 and $r = +7$, therefore the two quadratics are y^2 $- 7 = 0$, or $y = -2 \pm \sqrt{-3}$, and $y^2 - 4y + 3$ r $y = 2 \pm 1 = 3$ or 1, consequently $x = y+1 = -$ $\sqrt{-3}$, 4, and 2.

Find the value of x in the following equations.

EQUATIONS.	ANSWERS.
$- 2x^3 - x + 2 = 0.$	$x = 1, 2, -\frac{1}{2} + \sqrt{-\frac{3}{4}}$ and $-\frac{1}{2} - \sqrt{-\frac{3}{4}}$.
$- 2x^3 - 7x^2 - 8x + 12 = 0.$	$x = 1, 2, -2$, and -3.
$- 8x^3 + 9x^2 - 8x + 1 = 0.$	$x = \frac{7+\sqrt{45}}{2}, \frac{7-\sqrt{45}}{2}, \frac{1+\sqrt{-3}}{2}$, & $\frac{1-\sqrt{-3}}{2}$.
$18x^3 + 98x^2 + 162x + 81 = 0.$	$x = -1, -9, -4 + \sqrt{7}$, and $-4 - \sqrt{7}$.
$- 3x^2 - 4x - 3 = 0.$	$x = \frac{-1 \pm \sqrt{3}}{2}$ and $\frac{1 \pm \sqrt{3}}{2}$.
$\cdot 27x^2 - 14x + 120 = 0.$	$x = 2, 5, -3, \& -4.$
$- 3x^2 + 6x + 8 = 0.$	$x = -1, -2$, and $\frac{3 \pm \sqrt{-7}}{2}$.

8. $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0$. $x = 3 \pm \sqrt{5}$, and $1 \pm \sqrt{3}$.

9. $x^4 - 11x^3 + 41x^2 - 61x + 30 = 0$. $x = 1, 3, 5$, and 2.

RESOLUTION OF EQUATIONS BY APPROXIMATION, OR THE METHOD OF SUCCESSIVE SUBSTITUTION.

First Method. Find by trials the nearest integral value r of the root x, and substitute for x its equal $r \pm y$ in the given equation, and a new equation will arise involving only y and known quantities; then since y is a fraction, its square and higher powers are small when compared with itself, and may therefore be expunged from the equation, which will leave a simple equation, whence an approximate value of y may be easily obtained, and consequently a nearer value of the root. By substituting this value of r in the simple equation, another value of y will be found, which will give a still nearer value of the root, and so on, to any degree of accuracy that may be required.

Required the value of x in the equation $x^3 - 15x^2 + 63x - 50 = 0$.

By a few trials we find that x lies between 1 and 2, but nearer to 1. Let therefore $1 = r$ and $x = r + y$.

$$\text{Then} \left\{\begin{array}{rcl} x^3 &=& r^3 + 3r^2y + 3ry^2 + y^3 \\ -15x^2 &=& -15r^2 - 30ry - 15y^2 \\ 63x &=& 63r + 63y \\ -50 &=& -50 \end{array}\right\} = 0.$$

And by expunging the terms y^3, $3ry^2$, $15y^2$, we have $r^3 - 15r^2 + 63r + 3r^2y - 30ry + 63y - 50 = 0$; therefore

$$y = \frac{50 - r^3 + 15r^2 - 63r}{3r^2 - 30r + 63} = \frac{50 - 1 + 15 - 63}{3 - 30 + 63} = \frac{1}{36} = \cdot 027,$$

and $x = 1\cdot 027$ nearly.

Now, substituting $1\cdot 027$ for r in the last equation, we obtain

$$y = \frac{50 - 1\cdot 0832 + 15\cdot 8209 - 64\cdot 701}{3\cdot 1642 - 30\cdot 81 + 63} = \frac{\cdot 0367}{35\cdot 3542} = \cdot 00103, \text{ and}$$

$\therefore$ $1\cdot 027 + \cdot 00103 = 1\cdot 02803$, a still nearer value of x. Again, substituting $1\cdot 028$ for r, we have

$$y = \frac{50 - 1\cdot 086373952 + 15\cdot 85176 - 64\cdot 764}{3\cdot 170352 - 30\cdot 84 + 63} = \frac{\cdot 001386048}{35\cdot 330352} =$$

$\cdot 000039231$; consequently $x = 1\cdot 028039231$, which is true to the ninth place of decimals.

Second Method. Assume two numbers, differing only by unity in the last figure, as near the root as possible, and substitute them separately in the given equation instead of the unknown quantity; then collect the terms according to their signs, and mark the errors when in excess +, and when in defect —. Multiply the less error by the difference between the assumed numbers, and divide the product by the sum of the errors when they are unlike, but by their difference when they are alike. Add the quotient to the assumed number, whose error was multiplied when the assumed number is too small, otherwise subtract it, and the result will give the true root nearly.

To obtain the root still nearer, assume that last found, and another number differing from it only by unity in the last figure, and proceed with them in the same manner as before to get another correction, and so on, as far as is necessary.

Required the value of x in the equation $x^3 - 15x^2 + 63x - 50 = 0$.

Assume 1 and 1·1 as the trial numbers; then

1st Sup.		2d Sup.
1	x	1·1
63	$63x$	69·3
− 15	$-15x^2$	— 18·15
1	x^3	1·331
49	sums	52·481
50		50·
− 1	errors	+ 2·481

·1

2·481 1

3·481)·10000(·03 cor.

1·00

Hence x nearly = 1·03

Again, assume 1·03 and 1·02 as the trial numbers; then

1st Sup.		2d Sup.
1·03	x	1·02
64·89	$63x$	64·26
—15·9135	$-15x^2$	— 15·6060
1·092727	x^3	1·061208
50·069227	sums	49·715208
50		50
+ ·069227	errors	— ·284792

·069227

— ·284792 ·01

·354019)·00069227(·00196 correct.

Hence x = 1·03 — ·00196 = 1·02804, still more nearly.

Lastly, assume 1·02804 and 1·02803 as the trial numbers; then

1st Sup.		2d Sup.
1·02804	x	1·02803
64·76652	$63x$	64·76589
— 15·8529936240	$-15x^2$	—15·8526852135
1·0865007710	x^3	1·0864690653
50·0000271470	sums	49·9996738518
50·		50·
+ ·0000271470	errors	— ·0003261482

·0000271470

— ·0003261482 ·00001 1·02804

·0003532952)·000000000271470(·0000007684796

Hence x very nearly = 1·0280392315204

When one of the roots of an equation has been thus found, the rest may be found, thus:

Divide the given equation by x *minus* the root found, and the quotient will be an equation depressed a degree lower; then find a root of this new equation, and the number thus obtained will be a second root of the given equation.

Depress the second equation a degree lower by dividing it by x *minus* the root last found, and then find a third root, and so on, till the equation is reduced to a quadratic, the two roots of which, with those before found, will be all the roots of the original equation.

Thus in the equation $x^3 - 15x^2 + 63x - 50 = 0$, we found, by the second operation, one of the roots $= 1{\cdot}02804$, hence $x - 1{\cdot}02804)\ x^3 - 15x^2 + 63x - 50\ (x^2 - 13{\cdot}97169x + 48{\cdot}63627 = 0$, and the two roots of this quadratic when resolved in the usual way, are found to be $6{\cdot}57653$ and $7{\cdot}39543$, which are also roots of the given equation.

As formerly remarked, when the coefficient of the highest term is 1, the sum of all the roots is equal to the coefficient of the second term, and therefore we find that $1{\cdot}02804 + 6{\cdot}57653 + 7{\cdot}39543 = 15$. The sum of the product of the roots taken two by two is equal to the coefficient of the third term. Hence $7{\cdot}39543 \times 6{\cdot}57653 + 7{\cdot}39543 \times 1{\cdot}02804 + 6{\cdot}57653 \times 1{\cdot}02804 = 63$. And the continued product of all the roots is equal to the absolute term, thus:

$$7{\cdot}39543 \times 6{\cdot}57653 \times 1{\cdot}02804 = 50.$$

The root of a cubic equation, after the first figure is found by trial, may be readily found by extracting the cube root of the absolute term, taking care to add the coefficient of x according to its sign to the first and second trial divisors, thus:

Let $x^3 + 225x - 750 = 0$, here the root is soon found to be between 3 and 4, therefore

	750	)3·1891712 = x
+ 225) × 3 =	702	6
‹3+225=252	48000	91 × 1 = 91
91		2
25291 × 1 =	25291	938 × 8 = 7504
1		16
25383	22709000	9549 × 9 = 85941
7504		18
2545804 × 8 =	20366432	95671 × 1 = 95671
64		2
2553372	2342568000	956737 × 7 = 6697159
85941		14
255423141 × 9 =	2298808269	9567511 × 1 = 9567511
81		
255509163	43759731000	
95671		
25551011971 × 1 =	25551011971	
1		
25551107643	18208719029000	
6697159		
2555117461459×7=	17885822230213	
49		
2555124158667	322896798787000	
9567511		
255512425434211×1=	255512425434211	
	67384373352789000	

Let $x^3 - 7x + 7 = 0$. Here the root is between 1 and therefore

	−7	(1·35689655 = x
$(1^2 - 7) \times 1$ =	−6	2
× 3 − 7 = − 4	−1000	33 × 3 = 99
+ 99		6
−301 × 3 =	− 903	395 × 5 = 1975
9		10
−193	−97000	4056 × 6 = 24336
+ 1975		12
−17325 × 5 =	−86625	40688 × 8 = 325504
+ 25		
−15325	−10375000	
+ 24336		
−1508164×6 =	−9048984	
+ 36		
−1483792	−1826016000	
+ 325504		
−148053696×8=	−1184429568	
	141586432000	

1. Given $x^3 - 2x - 5 = 0$, to find an approximate value of x. Ans. $x = 2{\cdot}09455148$.

2. Given $x^3 - 7x + 7 = 0$, to find an approximate value of x. Ans. $x = 1{\cdot}35689655$.

3. Given $x^4 - 16x^3 + 40x^2 - 30x + 1 = 0$, to find an approximate value of x. Ans. $x = 13{\cdot}12488$.

4. Given $x^3 - 17x^2 + 54x - 350 = 0$, to find an approximate value of x. Ans. $x = 14{\cdot}954067$.

5. Given $x^4 - 3x^2 + 75x - 10000 = 0$, to find an approximate value of x. Ans. $x = 9{\cdot}8860027$.

6. Given $\sqrt{144x^2 - (x^2+20)^2} + \sqrt{196x^2 - (x^2+24)^2} - 114 = 0$, to find an approximate value of x. Ans. $x = 7{\cdot}123883$.

7. Given $x^3 + 438x^2 - 7825x - 98508430 = 0$, to find an approximate value of x. Ans. $x = 356{\cdot}9708968$.

8. Given $x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0$, to find an approximate value of x. Ans. $x = 12{\cdot}7564418$.

9. Given $x^3 - 12x + 8 = 0$, to find an approximate value of x. Ans. $x = 3{\cdot}0641776$.

10. Given $x^3 + 6{\cdot}7x^2 + 4{\cdot}5x - 10{\cdot}25 = 0$, to find an approximate value of x. Ans. $x = {\cdot}90018$.

11. Given $x^3 + 10x^2 + 50x - 2600 = 0$, to find an approximate value of x. Ans. $x = 10{\cdot}1794653$.

12. Given $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x - 54321$, to find an approximate value of x. Ans. $x = 8{\cdot}414455$.

OF EXPONENTIAL EQUATIONS.

Equations which contain quantities with unknown indices or exponents, as $a^x = b$, $a^{b^x} = c$, $x^x = a$, &c. are called Exponential or Transcendental Equations.

1. Given $a^x = b$ to find the value of x.

Since $a^x = b$, we have logarithm of (a^x) = logarithm of b, $\therefore\ x \log. a = \log. b$, or $x = \frac{\log. b}{\log. a}$; thus let $a = 8$, and $b = 100$; then $8^x = 100$ and $x = \frac{\log. 100}{\log. 8} = 2 \div {\cdot}903090 = 2{\cdot}2146187$.

2. Given $a^{b^x} = c$ to find the value of x.

NOTE. An exponential of this form means a to the power of b^x, and not a^b to the power x, which would then be expressed by a^{bx}.

Assume $z = b^x$ then $a^z = c$, and z log. $a =$ log. c, $\therefore z = \frac{\text{log. } c}{\text{log. } a} = b^x$. Again assume $y = \frac{\text{log. } c}{\text{log. } a}$; then $b^x = y$, and x log. $b =$ log. y, $\therefore x = \frac{\text{log. } y}{\text{log. } b}$. Thus let $a = 8$, $b = 2$, and $c = 200$, then $8^{2^x} = 200$; $\frac{\text{log. } c}{\text{log. } a} = \frac{\text{log. } 200}{\text{log. } 8} = \frac{2{\cdot}301030}{0{\cdot}903090} = 2{\cdot}548 = y$, and $x = \frac{\text{log. } y}{\text{log. } 2} = \frac{\text{log. } 2{\cdot}548}{\text{log. } 2} = \frac{{\cdot}406199}{{\cdot}301030} = 1{\cdot}349 = x$.

3. Exponentials of the form $x^x = a$ may be readily solved by the second method of approximation, page 131. The assumed numbers being substituted for x in the equation x log. $x =$ log. a, and the operation repeated a sufficient number of times, x may be obtained to any degree of exactness.

Thus let $x^x = 100$, to find an approximate value of x.

Here we have x log. $x =$ log. $100 = 2$, and it is obvious that x lies between 3 and 4, but nearer to 4. Assume, therefore, $x = 3{\cdot}5$ and $3{\cdot}6$;

Then $3{\cdot}5$ log. $3{\cdot}5 = 3{\cdot}5 \times {\cdot}544068 = 1{\cdot}904233 =$ 1st result,
and $3{\cdot}6$ log. $3{\cdot}6 = 3{\cdot}6 \times {\cdot}556303 = 2{\cdot}002691 =$ 2d result.
Whence $1{\cdot}904233 - 2 = \quad - {\cdot}095762 =$ 1st error,
and $2{\cdot}002691 - 2 = \quad + {\cdot}002691 =$ 2d error.
Sum of errors, ${\cdot}098453$

$\because {\cdot}002691 \div {\cdot}098453 = {\cdot}00273 =$ first correction, and $3{\cdot}6 - {\cdot}00273 = 3{\cdot}59727 = x$ nearly.

Again, assuming $x = 3{\cdot}59727$ and $3{\cdot}59728$, and using a table of logarithms to seven places, we have

$3{\cdot}59727$ log. $3{\cdot}59727 = 3{\cdot}59727 \times {\cdot}5559731 = 1{\cdot}9999854$
$3{\cdot}59728$ log. $3{\cdot}59728 = 3{\cdot}59728 \times {\cdot}5559743 = 1{\cdot}9999953$
Whence $1{\cdot}9999854 - 2 = \quad - {\cdot}0000146$ 1st error,
and $1{\cdot}9999953 - 2 = \quad - {\cdot}0000047$ 2d error.
Difference of errors, ${\cdot}0000099$;

$\therefore {\cdot}000000000047 \div {\cdot}0000099 = {\cdot}00000474747 =$ second correction, which added to $3{\cdot}59728$, gives $x = 3{\cdot}59728474747$ very nearly.

EQUATIONS.	ANSWERS.
1. Given $16^x = 200$.	$x = 1{\cdot}91096$.
2. Given $6^x = 1500$.	$x = 4{\cdot}081587$.
3. Given $6^{5^x} = 3000$.	$x = 1{\cdot}22707$.
4. Given $12^{4^x} = 6500$.	$x = {\cdot}910447$.
5. Given $x^x = 2000$.	$x = 4{\cdot}8278226$.
6. Given $x^x = 50$.	$x = 3{\cdot}28726192$.
7. Given $(5x)^x = 80$.	$x = 1{\cdot}9320805$.

PRACTICAL EXERCISES.

1. To find two numbers such that their sum shall be equal to the square of the greater, and their difference equal to the square of the less. Ans. 1·543688 and ·839286.

2. To find two numbers such that their sum shall be equal to the greater divided by the less, and their difference equal to the less divided by the greater.

Ans. 1·191486 and ·647799.

3. Given the sum of three numbers, 30, their product 780, and the sum of their squares 338, to find the number.

Ans. 5, 12, and 13.

4. Divide 20 into two parts, such that the cube of the greater shall be three times the cube of the less.

Ans. 11·8108288, and 8·1891712.

5. What two numbers are those whose sum is 24, and the difference of their cubes 866? Ans. 11 and 13.

6. Divide the number 20 into two parts, such that the square of the one shall be equal to the cube of the other.

Ans. 14·149 and 5·851.

7. The product of the ages of a man, his wife, and his son, is 12,000, the wife is 20 years older than the son, and the father's age is equal to those of his wife and son together. What is the age of each?

Ans. Father, 40; wife, 30; and son, 10.

8. Find two numbers such that their product multiplied by the less shall be 441, and their difference multiplied by the greater shall be 18. Ans. 9 and 7.

9. Three farms contain altogether 584 acres; the number of acres in the first is the square root of that in the second, and the cube root of that in the third. How many acres are in each? Ans. 8 acres, 64 acres, and 512 acres.

10. Required two numbers, whose sum, product, and difference of their squares are all equal.

Ans. 1·618034, and 2·618034.

11. What number is that whose square multiplied by its fourth part shall produce 432? Ans. 12.

12. There is a number consisting of two digits, and the sum of the squares of the digits added to their product is equal to the number itself, and the difference of the squares of the digits added to half the product is equal to half the number. Required the number. Ans. 64.

13. To divide the number 100 twice, so that the greater part of the first division shall be the cube of the less part of the second division, and the greater part of the second division three times the less part of the first division.

Ans. 68·0274 and 95·9177, the two greater.

14. Required the side of a cube of which the sum of the surface and solidity is 1600. Ans. 10.

15. What number is that of which its fourth power, divided by its half plus $14\frac{1}{4}$, is equal to 100? Ans. $3\frac{1}{2}$.

16. Find two numbers such that the sum of their squares divided by their product shall be equal to the less, and the difference of their cubes divided by their product shall be equal to the greater. Ans. 3·14789 and 2·14789.

17. Find two numbers such that their difference shall be equal to their product, and the difference of their squares shall be equal to their sum. Ans. 1·618 and 0·618.

18. A country girl exchanges geese for hens, giving two geese for three hens; the hens lay each one-half as many eggs as there were geese, and the girl sells nine eggs for as many pence as each hen laid eggs, and she receives in all 72d. How many geese did she exchange? Ans. 12.

19. Find three numbers, proportionals, whose sum is 38, and the sum of their squares is 532. Ans. 18, 12, 8.

20. What two numbers are those whose difference is 8, and whose product multiplied by their sum is 5040?

Ans. 10 and 18.

21. To find four numbers in continued proportion, so that the sum of the extremes shall be 27, and the difference of the means 6. Ans. 3, 6, 12, and 24.

22. To find four numbers in continued proportion, so that the product of the two least shall be 48, and the product of the two greatest 3888. Ans. 4, 12, 36, and 108.

23. There is a word of four letters, and the sum of the numbers of the two first letters, reckoning from the beginning of the alphabet, is equal to the sum of the two last, the product of the two first is 180, and that of the two last 110. and the sum of the numbers of the second and third is 37. Required the word. Ans. Love.

24. A company of merchants have a common stock of

£8240, of which each contributed 40 times as many pounds as there are partners, and they gain as much per cent. as there are partners, now on dividing the profit, after each had received ten times as many pounds as there are partners, there still remained £224. Required the number of partners.
Ans. 7, 8, or 10.

OF THE PROPERTIES OF NUMBERS.

Definitions. A prime number can only be divided by itself and unity. A composite number is composed of two or more factors neither of which is unity.

Proposition I. If $s =$ the sum, $d =$ the difference, and $p =$ the product of two numbers a and b, and n be any number;

then $a^n + b^n = s^n - nps^{n-2} + n \cdot \frac{n-3}{2} p^2 s^{n-4} - n \cdot \frac{n-4}{2} \cdot \frac{n-5}{3} p^3 s^{n-6} + n \cdot \frac{n-5}{2} \cdot \frac{n-6}{3} \cdot \frac{n-7}{4} p^4 s^{n-8} - n \cdot \frac{n-6}{2} \cdot \frac{n-7}{3} \cdot \frac{n-8}{4} \cdot \frac{n-9}{5} p^5 s^{n-10} + n \cdot \frac{n-7}{2} \cdot \frac{n-8}{3} \cdot \frac{n-9}{4} \cdot \frac{n-10}{5} \cdot \frac{n-11}{6} p^6 s^{n-12}$, &c.

And $a^n - b^n = d(s^{n-1} - \frac{n-2}{1} ps^{n-3} + \frac{n-3}{1} \cdot \frac{n-4}{2} p^2 s^{n-5} - \frac{n-4}{1} \cdot \frac{n-5}{2} \cdot \frac{n-6}{3} p^3 s^{n-7} + \frac{n-5}{1} \cdot \frac{n-6}{2} \cdot \frac{n-7}{3} \cdot \frac{n-8}{4} p^4 s^{n-9} -$, &c.)

Thus $a^2 + b^2 = s^2 - 2p$; $a^3 + b^3 = s^3 - 3ps$; $a^4 + b^4 = s^4 - 4ps^2 + 2p^2$; and $a^6 + b^6 = s^6 - 6ps^4 + 9p^2s^2 - 2p^3$.

And $a^2 - b^2 = ds$; $a^3 - b^3 = d(s^2 - p)$; $a^4 - b^4 = d(s^3 - 2ps)$; $a^5 - b^5 = d(s^4 - 3ps^2 + p^2)$; and $a^6 - b^6 = d(s^5 - 4ps^3 + 3p^2s)$.

PROP. II. The difference between two squares is equal to he sum of the roots added to twice the sum of the numbers etween the roots.

Let p and $p + n$ be the numbers or roots, the numbers etween them are $p + 1$, $p + 2$, $p + 3$, . . . $p + n - 3$, $p + n - 2$, $p + n - 1$, of which the number is $n - 1$, herefore their sum is $(p + 1 + p + n - 1)\ \frac{n-1}{2} = \frac{2p + n)\ n - 2p + n}{2}$, and its double, added to $2p + n$, the sum of the roots, gives $2pn + n^2 = (p + n)^2 - p^2$.

PROP. III. If r be any number, and n an integer, $\frac{r^n - 1}{r - 1}$ s an integer.

For if the division be performed, the quotient will be $r^{n-1} + r^{n-2}$ &c. $+ 1$ without a remainder.

Also, if n be an even number, $\frac{r^n - 1}{r + 1}$ is an integer, but if n e odd, $\frac{r^n + 1}{r + 1}$ is an integer.

NOTE. If r be the root of any arithmetical scale, and a, b, c, &c. ligits of which a number consists, that number will be represented y the series $a + br + cr^2 + dr^3$ &c. Thus, if the digits be 2, 3, 4, 5, the number in the decimal scale is $5 + 4 \times 10 + 3 \times 10^2 + 2 \times 10^3$, and in the duodecimal scale the number is $5 + 4 \times 12 + 3 \times 12^2 + 2 \times 12^3$.

PROP. IV. If from any number, of which r is the root of its cale, the sum of its digits be subtracted, the remainder will e divisible by $r - 1$.

For the remainder will be $b \times (r - 1) + c \times (r^2 - 1) + d \times (r^3 - 1)$ &c., each term of which is manifestly divisible y $r - 1$.

Cor. Hence, if the sum of the digits of any number be livisible by $r - 1$, the number will be divisible by $r - 1$, r any aliquot part of $r - 1$.

PROP. V. In any number, if the sum of the coefficients of he even powers of r be subtracted from the sum of the co-fficients of the odd powers, and the remainder be added to he number, the sum will be divisible by $r + 1$.

For the sum will be, $b(r+1) + c(r^2-1) + d(r^3+1) + e(r^4-1)$ &c. But when n is odd, $\frac{r^n+1}{r+1}$ is an integer, and when n is even, $\frac{r^n-1}{r+1}$ is an integer, therefore each of the terms is divisible by $r+1$.

Cor. 1. Hence, if the difference between the sum of the even digits and that of the odd digits of any number is divisible by $r+1$, the number itself is divisible by $r+1$.

Cor. 2. If a number wants all the even or all the odd digits, and the sum of the digits is divisible by $r+1$, the number is divisible by $r+1$.

In the common scale where $r=10$, $r-1=9$, and $r+1=11$, if we take the number 7587 we find the sum of its digits 27 is divisible by $r-1=9$. Hence the number itself is divisible by 9, and if we take the number 75834, we find that the sum of its odd digits exceeds that of its even digits by 11. Hence the number is itself divisible by 11. The sum of the digits of 9080709 and of 708070 are divisible by 11; therefore these numbers are divisible by 11.

To find whether a number be divisible by any of the prime numbers, we have no general rule but by actual division; we however know by inspection when it is divisible by 2, 3, or 5; and since $7 \times 11 \times 13 = 1001 = r^3 + 1 = r^2 - (r-1) \times (r+1)$; it follows that 1000 divided by any of these numbers will have -1 for a remainder, $(1000)^2$ will leave $+1$, $(1000)^3$ will leave -1, and so on: Therefore if we divide the number into periods of three figures from the right hand to the left, and take the sum of the 1st, 3d, &c. periods, and also the sum of the 2d, 4th, &c. periods, and subtract the latter sum from the former, then if the remainder is divisible by 7, 11, or 13, the given number will also be so, otherwise not.

Suppose the number 22,473,809,514, then $514 + 473 = 987$, and $809 + 22 = 831$, and $987 - 831 = 156$, which is divisible by 13, but not divisible by 11 or 7, therefore the given number is divisible by 13, but not by 7 or 11.

Prop. VI. Every product is divisible by any number which divides one of its factors.

Let the product be abc, and let $\frac{a}{r} = s$, then $a = rs$, and $abc = r(sbc)$, which is manifestly divisible by r.

Prop. VII. If the numbers a and b are each divisible by n, then their sum $a+b$, and their difference $a-b$, or

any multiples of these, $ma + mb$, and $ma - mb$ are also divisible by n.

Let $a = pn$, and $b = qn$, then $a + b = pn + qn = (p + q)n$; and $a - b = pn - qn = (p - q)n$, &c.

Cor. If neither a nor b is divisible by n, then neither $a+b$ nor $a - b$ is divisible by n.

PROP. VIII. If neither of the factors of a number ab are divisible by the prime number n, then the number ab is not divisible by n.

For if $a > n$, let $a = mn + r$ where $r < n$, but not $= 0$ nor $= 1$ (for then n would be divisible by r, and would not be a prime number), and $b = pn + s$, where $s < n$, but not $= 0$, nor $= 1$, then $ab = mpn^2 + mns + pnr + rs$, the first three terms of which are divisible by n. Hence if ab is divisible by n, then rs is also divisible by n. Let $n = qr + r'$ where $r' < r$, but not $= 0$ nor $= 1$, then $ns = qrs + r's$, but qrs being a multiple of rs is supposed to be divisible by n, therefore $r's$ is also divisible by n. In the same manner we find $r'' < r'$, but not $= 0$, such that $r''s$ is divisible by n, and so on, but as r consists of fewer units than n, and it decreases continually by one or more units, it will at last be equal to 1, and then $1s$ would be divisible by n, though it is less than n, which is impossible, wherefore ab is not divisible by n.

Cor. Hence, if n is not a divisor of a, neither is it a divisor of a^n, nor of $\frac{a^n}{b^n}$, for if $\frac{a^n}{b^n} = mn$, then $mnb^n = a^n$.

PROP. IX. Any number N, which is not a prime number, may be represented by several prime numbers raised to some powers, as $N = a^m b^n c^r$, and these numbers are obtained by dividing N, first by 2 as often as possible, the number of divisors is m; then divide the last remainder by 3 as often as possible, the number of divisors is n; then by 5 as often as possible, the number of divisors is r, &c.

When a number N is represented by $a^m b^n c^r$, &c. it will have for divisors $(1 + a + a^2 \ldots + a^m)(1 + b + b^2 \ldots + b^n) \times (1 + c + c^2 \ldots + c^r)$, &c. $= (m+1)(n+1)(r+1)$; whence it is manifest that N can be the product of two factors $\frac{1}{2}(m+1)(n+1)$, to $(r+1)$ ways, because these factors are two of the divisors.

If N is a square, m, n, r, are even numbers, and $\frac{1}{2}(m+1)(n+1)(r+1)$ will contain $\frac{1}{2}$, for which take 1.

If we wish the factors to be prime to each other, we do not use exponents, but consider $N = abcde$ &c., and if p represent the number of factors, 2^{p-1} is the number of ways in which N can be produced from two prime factors.

PROP. X. To find a number that shall have as many prime divisors as we choose.

Suppose 36 divisors, we must resolve 36 into its factors, either prime or not, as 4, 3, 3; then diminish each of them by unity and we get 3, 2, 2, for m, n, r; therefore $a^3b^2c^2$, is the form of a number which has 36 divisors; the most simple of such numbers is $2^3 \times 3^2 \times 5^2 = 1800$.

PROP. XI. Every odd number may be represented by $2x \pm 1$, or in relation to 6, it may be expressed by $6x \pm 1$, and $6x + 3$, and these expressions include all prime numbers except 2 and 3; but $6x + 3$ is not prime, and $6x \pm 1$ is not always prime when $x = 4, 6,$ &c.

PROP. XII. There is no algebraic formula which represents only prime numbers.

Suppose $p = ax^3 + bx^2 + cx + d$, a prime number, when $x = k$, then if $x = k + py$, y being an integer, we will have $p = (3ak^2 + 2bk + c)\,py + (3ak + b)\,p^2y^2 + ap^3y^3$, a prime number, which is impossible, for it is divisible by p, and is different from the other value of p.

Fermat says that the form $2^x + 1$ gives always primes if we take $x = 1, 2, 4, 6, 8, 16,$ &c.; but Euler shows that if $x = 32$, then $2^x + 1 = 641 \times 6700417$.

It is manifest that the number of primes is infinite, since the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11}$, &c. is infinite.

Though we cannot find an algebraic formula which contains only primes, nor one which contains all such numbers, or is their general law, yet we have some formulæ which are remarkable for the multitude of primes which they contain. Thus, if in the formula $x^2 + x + 41$, we take $x = 0, 1, 2, 3, 4,$ &c. we shall have 41, 43, 47, 53, &c. of which the first 41 terms are primes; in the formula $x^2 + x + 17$ the first 17 terms are primes; and in the formula $2x^2 + 29$, the first 29 terms are primes.

The expression $2x + 1$, according as x is even or odd, con-

tains the two forms $4x+1$ and $4x-1$, two principal divisions of prime numbers; the form $4x+1$ may be subdivided into $8x+1$ and $8x-3$, and the form $4x-1$, into $8x-1$ and $8x+3$, so that all the primes may be reduced to these four forms. The forms $6x \pm 1$ according as x is even or odd, give, in reference to 12, the four forms $12x+1$, $12x+5$, $12x-5$, and $12x-1$, each of which contains an infinite number of primes.

In general, if a is any number, every odd number may be represented by $4ax \pm b$ in which b is odd, and less than $2a$, and if from among all the possible values of b we expunge those which have a common measure with a, the remaining $4ax \pm b$ will contain all the prime numbers, divided in respect of the multiplier of $4a$, into as many forms as $\pm b$ shall have different values, and these forms altogether contain the whole of the prime numbers.

Suppose $-2a+b$, $-a+b$, b, $a+b$, $2a+b$, &c. to be an arithmetical progression, of which the general term is $ax'+b$, and suppose n a prime number which does not measure a, we can always find an infinite number of values of x, such that $\frac{ax+b}{n}$ shall be an integer.

For if x be taken successively $= 0, 1, 2, 3, 4, \ldots (n-1)$, the remainders of the divisions of $ax+b$ by n ought to be different one from another, and all less than n, and therefore one of them will be $= 0$. Two remainders cannot be the same, for if $ay+b$ and $az+b$ give the same remainder, y and z, being each less than n, their difference $a(y-z)$ would be divisible by n, but a is not divisible by n, and $y-z$ is less than n. Now if α be the least of the values of $\frac{ax+b}{n}$ and z be any integer, we shall have generally $x = \alpha + nz$, and thus the values of z which render $ax+b$ divisible by n, will themselves form an arithmetical progression α, $\alpha+n$, $\alpha+2n$, &c. of which the difference is n. Hence it follows that for n terms consecutively taken any where in the series $-a+b$, b, $a+b$, &c. there must be one divisible by n. In general the terms measured by n in the series of which we treat will be placed at the same distance the one from the other, and this distance comprehends always the number of terms n.

Suppose the series commences at the term $a+b$, when $x=1$, if we consider m consecutive terms ascending from the first, and from these m terms we reject all those which are divisible by n, there will remain $m\left(1-\frac{1}{n}\right)$ terms, not divisible by n.

In the applications of this formula the result will be always exact as often as m is a multiple of n, but m may be any number whatever,

not divisible by n, and then the number $m\left(1 - \frac{1}{n}\right)$ will contain an integer α, together with a remainder $\frac{\beta}{n}$, the integer has an unequivocal signification, as for the fraction $\frac{\beta}{n}$, it holds the place sometimes of 0 and sometimes of 1 according to different cases. This fraction expresses in some sort, the probability that the number of terms not divisible by n shall be $\alpha + 1$, but the number may be only α.

If ω is a prime number which is not a divisor of a, we can find in the same way, that in the finite series $a + b$, $2a + b$, $3a + b$, $ma + b$, there will be $m\left(1 - \frac{1}{\omega}\right)$ terms not divisible by ω, and the fraction that may be contained in $m\left(1 - \frac{1}{\omega}\right)$ will hold the place of the different cases of 0 and of 1.

Hence if we would know how many terms in the same series are not divisible by either ω or n we shall find this number to be $m\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{\omega}\right)$. If we suppose for greater simplicity that m is a multiple of $n\omega$, and make $m = m'n\omega$, we may distinguish in m four sorts of terms: 1st. N terms not divisible by either n or ω. 2d, $m'\omega$ terms not divisible by n. 3d, $m'n$ terms not divisible by ω; and 4th, m' terms not divisible by $n\omega$. Now it is evident that this last is comprised twice in the terms $m'\omega + m'n$, and that thus in reuniting all the distinct terms we shall have $\mathrm{N} + m'\omega + m'n - m'$, therefore $m = \mathrm{N} + m'\omega + m'n - m'$; hence $\mathrm{N} = m\left(1 - \frac{1}{n} - \frac{1}{\omega} + \frac{1}{n\omega}\right) = m \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{\omega}\right)$, a formula which is rigorously true, if m is a multiple of $n\omega$, and which approaches the truth so closely in all other cases, that the error can never amount to two units.

Similarly, it may be proved, that if n, o, p, q, &c. be primes, not divisors of a, the formula $m\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{o}\right)\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)$ &c. will represent the number of terms of the series $a + b$, $2a + b$, $3a + b$ &c. which are not divisible by any of the primes n, o, p, q, &c. This formula may in particular cases vary from the truth by reason of the fractions, introduced by each denominator, but the error can never amount to as many units as there are denominators.

If a and b have a common measure, the formula $ax + b$ can contain no prime number, if this common divisor be not one.

Suppose a and b to be prime to one another, and that thus $ax + b$ may represent different primes, we wish to find how many of these numbers are in the progression $a + b$, $2a + b$, $3a + b$, &c., $na + b$.

According to the preceding formula, we form the product $n\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)$ $\left(\frac{6}{7}\right)\cdot\cdot\left(\frac{\omega-1}{\omega}\right)$, in which we include all the prime numbers 2, 3, 5, 7, &c. to the greatest ω, contained in $\sqrt{(na+b)}$, excepting only those which divide a. If $a+b$ be greater than $\sqrt{(na+b)}$ the preceding formula will be the number sought; but if $a+b$ be less than $\sqrt{(na+b)}$, we must add to this formula as many units, as there are primes, less than $\sqrt{(na+b)}$, in the proposed series $a+b$, $2a+b$, &c.

Ex. How many prime numbers are in the first 1000 terms of the series 49, 109, 169, 229, 289, 349, of which the general term is $60x-11$? Here the 1000th term is 59989, its square root is 244, and the next less prime is 241, besides 60 is divisible by 3 and by 5; hence we must take for divisors all the primes from 7 to 241 which gives, $1000\left(\frac{6}{7}\cdot\frac{10}{11}\cdot\frac{12}{13}\cdot\frac{16}{17}\cdot\frac{18}{17}\cdot\cdot\cdot\cdot\frac{240}{241}\right)+2$. The two units are added because there are in the series two prime numbers 109 and 229 less than 241.

In order to accomplish a calculation of this kind, it will be useful to have a table of the values of $\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot$ &c., terminating at each of these numbers. The product to 241 is 0·201455, this divided by $\frac{2}{3}\times\frac{4}{5}$, which are not in the series, gives 0·377728, which multiplied by 1000 becomes 377·728, to which add 2, and it becomes 379·728, or 380, so that there are about 380 primes in 1000 terms of the series 49, 109, 169, &c.

Let it be required to find how many prime numbers are below 100,000? For this purpose, we consider the first 50,000 terms of the series 1, 3, 5, ... 99999. The square root of 99999 is nearly 316, and the next less prime is 313. Take, therefore, the product of $50000\left(\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{312}{313}\right)$, which will give $50000\times 0{\cdot}192686=9649$, to which we must add 66, because 313 is the 66th of the prime numbers, including 2, and it makes 9715 for the number of primes below 100000. Here the error cannot amount to 66, or $\frac{1}{165}$ of the whole.

Prop. XIII. If n is a prime number, and N any number not divisible by n, then the quantity $N^{n-1}-1$ is divisible by n, or N^n-N is divisible by n.

Since $(1+x)^n = 1 + nx + n \cdot \frac{n-1}{2} x^2 \ldots + x^n$, and all the terms of this development are divisible by n except the first, and the last, it is obvious, that $(1+x)^n$ when divided by n, will give the same remainder that x^n+1 will give, that is, in point of remainders, $(1+x)^n = x^n+1$, and supposing $1+x = \text{N}$ we have $\text{N}^n = (\text{N}-1)^n+1$, and subtracting N from each, we have $\text{N}^n - \text{N} = (\text{N}-1)^n - (\text{N}-1)$. As N may be any number, we may substitute $\text{N}-1$ instead of it, then $(\text{N}-1)^n - (\text{N}-1) = (\text{N}-2)^n - (\text{N}-2)$, and by diminishing the value in this way, we shall at last have on the right hand side of the equation $(\text{N}-\text{N})^n - (\text{N}-\text{N})$ which is evidently $= 0$; therefore $\text{N}^n - \text{N}$ divided by n gives 0 for a remainder, but $\text{N}^n - \text{N} = \text{N}(\text{N}^{n-1}-1)$, and N is not divisible by n, consequently $\text{N}^{n-1} - 1$ must be divisible by n.

Cor. When n is prime $x^{n-1} - 1$ is divisible by n, or $\frac{x^{n-1}-1}{n}$ is an integer, and x may be $= \pm 1, \pm 2$, &c., from $-\frac{1}{2}n$ to $+\frac{1}{2}n$, therefore the number of solutions is $n-1$, the exponent of x.

Prop. XIV. If n is a prime number, then the product $1 \cdot 2 \cdot 3 \cdot 4 \ldots (n-1)$ increased by 1, is divisible by n.

For putting $m = n-1$ we have by the theory of differences $1 \cdot 2 \cdot 3 \ldots m = m^m - m \cdot (m-1)^m + m\left(\frac{m-1}{2}\right) \cdot (m-2)^m - m + \frac{m-1}{2} \cdot \frac{m-2}{3}(m-3)^m$, &c.; where, if we neglect the multiples of n, we shall have by the preceding theorem, $m^m = 1, (m-1)^m = 1, (m-2)^m = 1$, &c.; therefore the product $1 \cdot 2 \cdot 3 \ldots m$ is reduced to $1 - m + m \cdot \left(\frac{m-1}{2}\right) -$, &c., and the number of terms is m; but this is the development of $(1-1)^m$ except its last term, which is $+1$, since m is even, therefore the sum of the terms, in respect of the remainders, is $(1-1)^m - 1$, which is manifestly $= -1$.

This theorem is remarkable, as it is true only when n is prime, for if we take $n =$ the product of two factors, these factors are also terms of the product $1 \cdot 2 \cdot 3 \ldots (n-1)$, and

will consequently divide it, so that if 1 be added to the product, the remainder will be $+1$; the same is the case when $n = a^2$, for both a and $2a$ are factors of the product $1 . 2 . 3 \ldots (n-1)$, which is therefore divisible by a^2 or n.

This is a certain, but very tedious, method of finding whether or not n is a prime number.

NOTE. The numbers $n-1$, $n-2$, $n-3$, &c. considered as the remainders of the division by n, are equivalent to the remainders $-1, -2, -3$, &c.; besides n being odd, the number of factors, $1 . 2 . 3 \ldots (n-1)$ will be even, therefore the product divided by n, will leave the same remainder as $\pm 1^2 . 2^2 . 3^2 \ldots \left(\frac{n-1^2}{2}\right)$, the sign being $+$ when n is of the form $4x+1$, and $-$ when it is of the form $4x-1$.

Hence 1. If a prime number n, be of the form $4x+1$, the product $1 . 2 . 3 \ldots \left(\frac{n-1}{2}\right)^2 + 1$ will be divisible by n; thus we know the sum of two squares a^2+1, of which n is a divisor.

2. If the prime number n, be of the form $4x-1$, the product $1 . 2 . 3 \ldots \left(\frac{n-1}{2}\right)^2 - 1$ will be divisible by n, consequently n will divide one or other of the two quantities, $1 . 2 . 3 \ldots \left(\frac{n-1}{2}\right) + 1$, or $1 . 2 . 3 \ldots \left(\frac{n-1}{2}\right) - 1$.

PROP. XV. If we try the division of any number N by all the prime numbers, 2, 3, 5, 7, &c. as far as $\sqrt{N}$, and find that none of them will divide N without a remainder, then N is a prime number.

For if N could be divided by a number greater than $\sqrt{N}$, the quotient would be less than $\sqrt{N}$, and so a number less than $\sqrt{N}$ would divide it, which is impossible.

PROP. XVI. If n be a prime number, and the number N a polynomial, of the form $ax^m + bx^{m-1} + cx^{m-2}$, &c., then there cannot be more than m values of x, between $+\frac{1}{2}n$ and $-\frac{1}{2}n$, which make N divisible by n.

Let p be the first value of x which renders N divisible by n, we may make $N = (x-p) N' + An$, and we shall have N' a polynomial of the form $ax^{m-1} + bx^{m-2} + cx^{m-3} +$, &c.

Let p' be a second value of x, which renders N divisible by n, this must also render $(x - p)$ N$'$ divisible by n, since p and p' are each less than $\frac{1}{2}n$; therefore if N, a polynomial of the mth degree, be a second time divisible by n, then also is N$'$ a polynomial of the degree $(m - 1)$ divisible by n, consequently, N admits of but one solution more than N$'$, hence there can be but m different values of x, between $+\frac{1}{2}n$ and $-\frac{1}{2}n$, which render N divisible by n.

If after one solution, as $x = p$, we take $x = p + nz$, then all the values of x which resolve this formula will also satisfy the formula $\frac{N}{n}$ an integer.

PROP. XVII. If n be a prime, and N a polynomial of the form $ax^m + bx^{m-1} + cx^{m-2} +$ &c., which is a divisor of $x^{n-1} - 1$, then there are always m values of x, between $+\frac{1}{2}n$, and $-\frac{1}{2}n$, which render N divisible by n.

Let $x^{n-1} - 1 = \text{NP}$, P being another polynomial of the form $ax^{n-(m+1)} + bx^{n-(m+2)} + cx^{n-(m+3)} +$ &c. Since there are $n - 1$ values of x, namely, $\pm 1, \pm 2, \pm 3, \pm 4 \ldots \pm \frac{n-1}{2}$, which render $x^{n-1} - 1$ divisible by n, each of these values ought to render N, or P, divisible by n; but among these $n - 1$ values, there are not more than m, which render N divisible by n, and there cannot be less than m values which do so, for then there would be more than $n - 1 - m$ values, which render P divisible by n, but this cannot be, as P is only of the degree $n - 1 - m$; therefore, the number of values of x which render N divisible by n and which lie between $-\frac{1}{2}n$, and $+\frac{1}{2}n$, are precisely m. The same would also be true, if N was a divisor of $x^{n-1} - 1 + n\text{R}$, R being $= ax^{n-s} + bx^{n-s-1} +$ &c.; or a polynomial of any degree less than n.

PROP. XVIII. If the prime number n is a divisor of the number $x^2 + \text{N}$, (N being any number positive or negative) then the number $(-\text{N})^{\frac{n-1}{2}} - 1$, is divisible by n, and reciprocally: if this condition be fulfilled, there is a number x less than $\frac{1}{2}n$, such, that $x^2 + \text{N}$ shall be divisible by n, (except when $n = 2$, and N is divisible by n).

For since $x^2 + \text{N}$ is divisible by n, the remainders of $\frac{x^2}{n}$

and of $-\frac{N}{n}$ must be equal, therefore in respect of remainders, $x^2 = -N$, consequently $x^{n-1} - 1 = (-N)^{\frac{n-1}{2}} - 1$, but the former is divisible by n, therefore the latter must also be divisible by n.

Again, if $(-N)^{\frac{n-1}{2}} - 1$, be divisible by n, let this quantity be nr, then $x^{n-1} - 1 - nr = x^{n-1} - (-N)^{\frac{n-1}{2}}$. Suppose $n - 1 = 2b$ and $-N = +M$, the second side becomes $x^{2b} + M^b$, which is obviously divisible by $x^2 - M$, $= x^2 + N$, consequently $x^2 + N$ must also divide $x^{n-1} - 1 - nr$, there are therefore two values of x, which differ only in their sign, less than $\frac{1}{2}n$, which render $x^2 + N$ divisible by n.

Cor. When N is not divisible by the prime n, then $N^{n-1} - 1$ is always divisible by n, the factors of this quantity being $N^{\frac{n-1}{2}} + 1$, and $N^{\frac{n-1}{2}} - 1$, consequently one of these must be divisible by n; therefore $N^{\frac{n-1}{2}}$ when divided by n, will have a remainder of either $+ 1$, or $- 1$.

OF CONTINUED FRACTIONS.

A CONTINUED fraction is one, of which the denominator is a mixed number, the fractional part of which has also a mixed number for its denominator, and so on.

In order to obtain continued fractions, we proceed in the same manner as in finding the greatest common measure of two numbers, and we afterwards make use of the quotients to form the continued fraction.

Let it be required to reduce the fraction $\frac{101}{59}$ into a continued fraction.

```
59)101(1
    42)59(1
       17)42(2
           8)17(2
              1)8(8
```

Here the quotients are 1, 1, 2, 2, 8, and since the given fraction is an improper one, we have 1 for the first term of

the continued fraction, and taking the other quotients for the denominators of the continued fraction sought, and 1 for each of the numerators, we have,

therefore, $\frac{101}{59} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{8}}}}$

Here, we first divide 101 by 59, and the quotient is $1 + \frac{42}{59}$, but as $\frac{42}{59}$ has not 1 for its numerator, we divide by 42 and it gives $1 + \frac{17}{42}$; we now divide by 17, and obtain $2 + \frac{8}{17}$; lastly, dividing by 8, we get $2 + \frac{1}{8}$, and this completes the fraction.

When any denominator is divided by its numerator, the mixed number resulting is called the *complete quotient.* Thus $1 + \frac{17}{42}$, $2 + \frac{8}{17}$, $2 + \frac{1}{8}$, are complete quotients.

The integral part of the complete quotient is called the *partial quotient.*

Let it be required to reduce $\frac{77}{183}$ to a continued fraction.

```
77)183(2
   29)77(2
      19)29(1
         10)19(1
             9)10(1
                1)9(9
```

Since the given fraction is a proper one, the first term of the continued fraction is 0; hence

$$\frac{77}{183} = 0 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{9}}}}}}$$

Reduce $\frac{27}{53}, \frac{89}{139}, \frac{479}{1728}, \frac{597}{3796}$ to continued fractions.

In order to convert continued into converging fractions, place the integer, and the partial quotients in their order, in one line. Below the integer write 1 with 0 for the denominator, below the first partial quotient place the integer with 1 for its denominator, then multiply both the numerator and denominator by the figure above it, and to the products add the preceding numerator and denominator, and set the resulting fraction below the next partial quotient, and so on. Thus, taking the first example in the preceding page, the partial quotients were

$$\begin{array}{lllll} 1, & 1, & 2, & 2, & 8. \\ \frac{1}{0} & \frac{1\cdot1+1}{1\cdot1+0}=\frac{2}{1}, & \frac{2\cdot2+1}{1\cdot2+1}=\frac{5}{3}, & \frac{5\cdot2+2}{3\cdot2+1}=\frac{12}{7}, & \frac{12\cdot8+5}{7\cdot8+3}=\frac{101}{59}. \end{array}$$

If the common fraction be a proper one, as $\frac{89}{237}$, where the partial quotients are 2, 1, 1, 1, 29, we must write $\frac{0}{1}$ below the first quotient, and 1 for the numerator, and the first partial quotient for the denominator of the next fraction. Thus,

$$\begin{array}{lllll} 2, & 1, & 1, & 1, & 29. \\ \frac{0}{1} & \frac{1}{2}, \frac{1\cdot1+0}{2\cdot1+1}=\frac{1}{3}, & \frac{1\cdot1+1}{3\cdot1+2}=\frac{2}{5}, & \frac{2\cdot1+1}{5\cdot1+3}=\frac{3}{8}, & \frac{3\cdot29+2}{8\cdot29+5}=\frac{89}{237}. \end{array}$$

The common fractions thus produced are called *Converging Fractions.*

The successive converging fractions are continual approximations to the value of the given fraction, each of them being nearer to the true value than any of the preceding fractions, and they are alternately greater and less than the given fraction. Thus, in the development of the fraction $\frac{101}{59}$, the first converging fraction $\frac{1}{1}=\frac{59}{59}$ is too little, and the second $\frac{2}{1}=\frac{118}{59}$ is too great, but nearer to $\frac{101}{59}$ than $\frac{59}{59}$. The next $\frac{5}{3}=\frac{295}{177}$ is less than $\frac{101}{59}=\frac{303}{177}$, but nearer than $\frac{2}{1}=\frac{354}{177}$, and so on.

These converging fractions have also the advantage of being in their lowest terms, and nearer than any intermediate fraction.

If any two adjacent converging fractions be taken, and the numerator of each be multiplied by the denominator of the other, the difference of the products is always $= 1$, namely, $+ 1$ when the greater fraction is in the 3d, 5th, or any odd rank, ($\frac{1}{0}$, or $\frac{0}{1}$, being supposed the first) and $- 1$ when it is in the 2d, 4th, or any even rank.

Thus, in the development of $\frac{101}{59}$, $\frac{2}{1}$ is in the third rank, and $\frac{2}{1} \times 1 - \frac{1}{1} \times 1 = +1$; again, $\frac{5}{3}$ is in the 4th rank, and $\frac{5}{3} \times 1 - \frac{2}{1} \times 3 = -1$; also, $\frac{12}{7} \times 3 - \frac{5}{3} \times 7 = +1$, and $\frac{101}{59} \times 7 - \frac{12}{7} \times 59 = -1$.

Convert $\frac{23}{41}$, $\frac{29}{54}$, $\frac{65}{113}$, and $\frac{419}{1237}$ into continued fractions, and thence into a series of converging fractions.

Ans. Convergents, $\frac{1}{1}, \frac{1}{2}, \frac{4}{7}, \frac{5}{9}, \frac{9}{16}, \frac{23}{41}$.

.. .. $\frac{1}{1}, \frac{1}{2}, \frac{7}{13}, \frac{29}{54}$.

.. .. $\frac{1}{1}, \frac{1}{2}, \frac{3}{5}, \frac{4}{7}, \frac{19}{33}, \frac{65}{113}$.

.. .. $\frac{1}{2}, \frac{1}{3}, \frac{20}{59}, \frac{21}{62}, \frac{419}{1237}$.

USES OF CONTINUED FRACTIONS.

I. To express in lower terms the approximate value of any given fraction.

Suppose the tropical year to be 365 days, 5 hours, 48 minutes, 48 seconds, or 5 hours, 48 minutes, 48 seconds, $=$ 20928 seconds, more than the civil year of 365 days, required the intercalations necessary to make them agree?

The day, or 24 hours = 86400 seconds, therefore the fraction to be reduced is, $\frac{86400}{20928} = \frac{450}{109}$, and in finding the common measure of this, the quotients are 4, 7, 1, 3, 1, 2; hence, when reduced into converging fractions, we have

$$\begin{array}{ccccccc} & 4, & 7, & 1, & 3, & 1, & 2. \\ \frac{1}{0}, & \frac{4}{1}, & \frac{29}{7}, & \frac{33}{8}, & \frac{128}{31}, & \frac{161}{39}, & \frac{450}{109}. \end{array}$$

Whence it is evident that, to make them agree completely, there must be 109 intercalations, or additions of a day, in 450 years; but we may approximate to this, first by adding 1 day in 4 years, or more nearly, 7 days in 29 years, or still more nearly and conveniently, by adding 8 days in 33 years, adding the eighth day at the end of 5 years instead of 4, and so on.*

Required the approximate ratio of the circumference of a circle to its diameter, to 10 places of decimals.

Here, the fraction to be reduced is $\frac{31415926536}{10000000000} = \frac{3926990817}{1250000000}$, and the quotients are 3, 7, 15, 1, 292, 1, &c.

Hence, the converging fractions are,

$$\begin{array}{ccccccc} & 3, & 7, & 15, & 1, & 292, & 1. \\ \frac{1}{0}, & \frac{3}{1}, & \frac{22}{7}, & \frac{333}{106}, & \frac{355}{113}, & \frac{103993}{33102}, & \frac{104348}{33215}, \end{array}$$

&c. which are alternately greater and less than the circumference divided by the diameter, and are each expressed in their lowest terms; the second, $\frac{22}{7}$, is the ratio assigned by Archimedes, and the fourth is that given by Metius, which, of course, is more accurate than the former.

If the operation be carried farther, the converging fractions will continually approach nearer to the true ratio, and they will become the more intricate the farther they are extended.

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* $\frac{450}{109}$, or $\frac{900}{218}$ gives 24 days in 100 years, and 2 days over. Hence we may intercalate 1 day every 4 years, but make every century a common year, with the exception of every fifth and fourth century alternately, which will make up the 2 days in 900 years.

II. To approximate to the roots of numbers.

Required the square root of 2?

Here the integer is 1. Making $\sqrt{2} = 1 + \frac{1}{x}$; then $x = \frac{1}{\sqrt{2}-1}$. Multiply both terms by $\sqrt{2}+1$, in order to render the denominator rational, and it becomes $x = \frac{\sqrt{2}+1}{1} = 2 + \frac{\sqrt{2}-1}{1} = 2 + \frac{1}{x}$; therefore the quotient is continually 2.

Hence

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + 1 \text{ \&c.}}}}}$$

Where the law of continuation is evident, all the denominators being 2.

Quotients, 1, 2, 2, 2, 2, 2, &c.

Convergents, $\frac{1}{0}, \frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$, &c.

Required the square root of 11?

$\sqrt{11} = 3 + \frac{1}{x}$, and $x = \frac{1}{\sqrt{11}-3} = \frac{\sqrt{11}+3}{2} = 3 + \frac{\sqrt{11}-3}{2}$

$= 3 + \frac{1}{y}$, then

$$y = \frac{2}{\sqrt{11}-3} = \frac{2(\sqrt{11}+3)}{2} = \frac{\sqrt{11}+3}{1} = 6 + \frac{\sqrt{11}-3}{1} =$$

$6 + \frac{1}{z}$, and

$$z = \frac{1}{\sqrt{11}-3} = \frac{\sqrt{11}+3}{2} = x, \text{ wherefore}$$

$$\sqrt{11} = 3 + \cfrac{1}{3 + \cfrac{1}{6 + \frac{1}{x}}}$$

where the denominator repeats 3 and 6, alternately.

Quotients, 3, 3, 6, 3, 6, 3, &c.

Convergents, $\frac{1}{0}, \frac{3}{1}, \frac{10}{3}, \frac{63}{19}, \frac{199}{60}, \frac{1257}{379}$, &c.

Required the square root of 13?

Ans. $$\sqrt{13} = 3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \frac{1}{x}}}}}}.$$

Here the quotients are $3, \dot{1}, 1, 1, 1, \dot{6}, 1$, &c.

Converging fractions, $\frac{1}{0}, \frac{3}{1}, \frac{4}{1}, \frac{7}{2}, \frac{11}{3}, \frac{18}{5}, \frac{119}{33}$, &c.

Required the square root of 35?

Ans. $$\sqrt{35} = 5 + \cfrac{1}{1 + \cfrac{1}{10 + \frac{1}{x}}}.$$

Here the quotients are $5, \dot{1}, \dot{10}, 1, 10, 1, 10$, &c.

Convergents, $\frac{1}{0}, \frac{5}{1}, \frac{6}{1}, \frac{65}{11}, \frac{71}{12}, \frac{766}{131}, \frac{837}{143}$, &c.

Required the square root of 19?

Ans. $$\sqrt{19} = 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{8 + \frac{1}{x}}}}}}}.$$

Here the quotients are $4, \dot{2}, 1, 3, 1, 2, \dot{8}, 2$, &c.

Converging fractions, $\frac{1}{0}, \frac{4}{1}, \frac{9}{2}, \frac{13}{3}, \frac{48}{11}, \frac{61}{14}, \frac{170}{39}, \frac{1421}{326}$, &c.

Sometimes the number of quotients in the circle will be very great. Thus in $\sqrt{211}$ they are 14, 1, 1, 9, 5, 1, 2, 2, 1, 1, 4, 3, 1, 1, 3, 1, 1, 3, 1, 3, 4, 1, 1, 2, 2, 1, 5, 9, 1, 1, 28.

It is worthy of remark, that the last quotient of the circle is always double of the integer.

In extracting the square or cube root, the partial quotients may be found from the preceding converging fractions. Thus, in taking the square root of 7, the integer is 2, whence set down the two first converging fractions.

$$2,\ 1,\ 1,\ 1,\ 4,\ 1.$$
$$\frac{1}{0},\ \frac{2}{1},\ \frac{3}{1},\ \frac{5}{2},\ \frac{8}{3},\ \frac{37}{14}.$$

Let the greater fraction $\frac{2}{1} = \frac{r}{r'}$, and the preceding $\frac{1}{0} = \frac{p}{p'}$, then the next quotient is got thus, let $P = \frac{2r}{r^2 - 7^* r'^2} = \frac{4}{3}$, then the nearest integer to $\frac{P - p'}{r}$ is the next quotient $=$ 1, whence the next converging fraction is $\frac{3}{1} = \frac{r}{r'}$, then $\frac{p}{p'} = \frac{2}{1}$, whence $P = \frac{2 \cdot 3}{9 - 7 \cdot 1} = \frac{6}{2}$, and $\frac{P - p'}{r} = 1$, and so on.

Let it be required to find the cube root of 17.

We know by inspection that the integer is 2; therefore the two first converging fractions are got, namely,

$$2,\ 1,\ 1,\ 3,\ 138,\ \&c.$$
$$\frac{1}{0},\ \frac{2}{1},\ \frac{3}{1},\ \frac{5}{2},\ \frac{18}{7},\ \&c.$$

Now, calling $\frac{2}{1} = \frac{r}{r'}$, and $\frac{1}{0} = \frac{p}{p'}$, then $P = \frac{3r^2}{r^3 - 17\dagger r'^3}$, $= \frac{3 \cdot 2^2}{2^3 - 17 \cdot 1^3} = \frac{12}{8 - 17} = -\frac{4}{3}$, and $\frac{P - p'}{r'} = \frac{4}{3}$, the nearest integer to which is 1, and this is the next partial quotient,

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* 7 is the given number whose root is to be found.
† 17 is the given number whose root is to be found.

from which the next converging fraction is $\frac{3}{1}$. Again, $\frac{r}{r'} = \frac{3}{1}, \frac{p}{p'} = \frac{2}{1}$, and $P = \frac{3 \cdot 3^2}{3^2 - 17 \cdot 1^2} = \frac{27}{27 - 17} = \frac{27}{10}$, and $\frac{P - p'}{r'} = \frac{2\frac{7}{10} - 1}{1} = 1$, the next quotient, whence the next converging fraction is $\frac{5}{2}$. Again, $\frac{r}{r'} = \frac{5}{2}, \frac{p}{p'} = \frac{3}{1}$, and $P = \frac{3 \cdot 5^2}{5^2 - 17 \cdot 2^2} = \frac{75}{125 - 136} = -\frac{75}{11}$, and $\frac{P - p'}{r'} = \frac{6\frac{9}{11} - 1}{2} = 3$, the next quotient, whence the next converging fraction is $\frac{18}{7}$; and, proceeding in this way, we find the succeeding quotients to be 138, 1, 1, 3, 2, 3, 1, 1, 47, 1, 2, 2, 22, 9, &c.

III. To approximate to the roots of quadratic equations.

Let $ax^2 + bx + c = 0$, be the equation; this, when resolved in the usual way, gives $x + \frac{b}{2a} = \pm \frac{\sqrt{(b^2 - 4ac)}}{2a}$. Hence we have only to approximate to the value of $\sqrt{(b^2 - 4ac)}$.

Let the equation be $x^2 - 7x - 7 = 0$.

Here $x - \frac{7}{2} = \pm \frac{\sqrt{(49 - 4 \cdot 7)}}{2} = \frac{\sqrt{21}}{2}$; therefore, all that is required is to approximate to the value of $\sqrt{21}$.

Here the partial quotients, after the integer 4, are 1, 1, 2, 1, 1, 8, which are constantly repeated, and the converging fractions are

$$\begin{array}{cccccccc} 4, & 1, & 1, & 2, & 1, & 1, & 8. & \\ \frac{1}{0}, & \frac{4}{1}, & \frac{5}{1}, & \frac{9}{2}, & \frac{23}{5}, & \frac{32}{7}, & \frac{55}{12}, & \frac{472}{103}. \end{array}$$

Therefore $x = \frac{1}{2}\left(7 \pm \frac{472}{103}\right) = \frac{721 \pm 472}{206} = 5 \cdot 7912$, or $1 \cdot 2087$.

1. Let the equation be $x^2 + 3x - 5 - 12 = 0$.
Ans. $x = 2 \cdot 8875$, or $- 5 \cdot 8874$.

2. Let the equation be $4x^2 - 15x + 13 = 0$.
Ans. $x = 2 \cdot 40533$, or $1 \cdot 34467$.

3. Let the equation be $x^2 + 6x - 37 = 0$.

Ans. $x = 3{\cdot}7823$, or $-9{\cdot}7823$.

4. Let the equation be $x^2 + 2x - 13 = 0$.

Ans. $x = 2{\cdot}7416$, or $-4{\cdot}7416$.

IV. To approximate to the roots of higher equations.

Assume $r =$ the nearest integer less than the value of x, make $x = r + \frac{1}{z}$, and substitute this value and its powers, instead of x and its powers, and we get a new equation, which, being cleared of fractions and its highest term made positive, will give an equation in which r is always greater than 1, and the nearest value of it, less than z, may be found as before. This being denominated r', and making $r = r' + \frac{1}{z'}$, and this value substituted, and the equation arranged as before, gives an equation for finding z', which will also be greater than 1; in this manner we may proceed to any length that may be required, and it is manifest that the values of z, z', &c. are partial quotients, from which the value of x may be obtained in continued or converging fractions.

Let the equation be $x^3 - x^2 - 2x + 1 = 0$. Here by substituting 1 for x the equation becomes $= -1$, and by substituting 2 for x it becomes $= +1$, therefore x is between 1 and 2. Let $x = 1 + \frac{1}{z}$, then

$$+x^3 = \quad 1 + \frac{3}{z} + \frac{3}{z^2} + \frac{1}{z^3}$$

$$-x^2 = -1 - \frac{2}{z} - \frac{1}{z^2}$$

$$-2x = -2 - \frac{2}{z}$$

$$+1 \;\; = +1$$

$$-1 - \frac{1}{z} + \frac{2}{z^2} + \frac{1}{z^3} = 0.$$ Multiply this by $-z^3$ and we have $z^3 + z^2 - 2z - 1 = 0$. In this equation we

find z between 1 and 2, and substituting $z = 1 + \frac{1}{z'}$, we get

$$+z^3 = +1 + \frac{3}{z'} + \frac{3}{z'^2} + \frac{1}{z'^3}$$

$$+z^2 = +1 + \frac{2}{z'} + \frac{1}{z'^2}$$

$$-2z = -2 - \frac{2}{z'}$$

$$-1 \;= -1$$

$-1 + \frac{3}{z'} + \frac{4}{z'^2} + \frac{1}{z'^3} = 0$. This multiplied by $-z'^3$ gives $z'^3 - 3z'^2 - 4z' - 1 = 0$ where z' is between 4 and 5, and so on.

We may remark that the terms of the transformed equations may be got from one another; thus the coefficient of the first term is the equation to be transformed, $z^3 + z^2 - 2z - 1$, the integral value of z being taken and substituted instead of z; the second term is got by multiplying each term of the first by the exponent of z in it, and then dividing the whole by z; thus,

$\frac{z^3 \times 3 + 2z \times 2 - 2z \times 1 + 1 \times 0}{z} = 3z^2 + 2z - 2$. The third term is got by multiplying each term of the second by the exponent of z in it, and then dividing by $2z$; thus,

$\frac{3z^2 \times 2 + 2z \times 1 - 2 \times 0}{2z} = 3z + 1$. The next is got by multiplying each term of the third by the exponent of z in it, and dividing by $3z$, and it becomes $+1$. After this, substitute the integral values of z, in the equation to be transformed, and it will give the coefficients of the transformed equation. Thus, by substituting 1 instead of z, the coefficient of the first term is $1^3 + 1^2 - 2 \times 1 - 1 = -1$, the second becomes $3 \cdot 1^2 + 2 \times 1 - 2 = +3$, and the third $3 \times 1 + 1 = 4$.

In like manner, the first term of the next transformation is $z'^3 - 3z'^2 - 4z' - 1$, the second $3z'^2 - 6z' - 4$, the third $3z' - 3$, and the fourth 1, and as $z' = 4$, the transformed equation becomes $z''^3 - 20z''^2 - 9z'' - 1 = 0$ where z'' is between 20 and 21. The terms of the next transformation are, in the same way, $z''^3 - 20z''^2 - 9 + 1$,

$3z''^2 - 40z'' - 9$, $3z'' - 20$, and $+1$, where, putting 20 instead of z'', we obtain the next transformed equation, namely, $181y^3 - 391y^2 - 40y - 1 = 0$. Here y is between 2 and 3. The next tranformed equation is therefore $197y'^3 - 568y'^2 - 695y' - 181 = 0$, where y' is between 3 and 4; and proceeding in this manner, we find the succeeding values to be 1, 6, 10, &c.

Hence the approximate values of x are

$$1, 1, 4, 20, 2, 3, 1, 6, 10, \&c.$$

$$\frac{1}{0}, \frac{1}{1}, \frac{2}{1}, \frac{9}{5}, \frac{182}{101}, \frac{373}{207}, \frac{1301}{722}, \frac{1674}{929}, \frac{11345}{6296}, \frac{115124}{63889}.$$

After finding several of the partial quotients in this way, we may find as many more from these; thus from the quotient 2 we find the converging fraction $\frac{373}{207}$, multiplying the denominator of the preceding 101 by the highest exponent of the equation -1 gives 202; place 207 for the denominator it becomes $\frac{202}{207}$; place also the coefficient of the second term of the transformed equation 568, for the numerator, and the coefficient of the first term, for the denominator, and give it the sign contrary to that of the second term, add these fractions, and develop the sum into a continued fraction, and it will give a number of partial quotients; thus, $\frac{202}{207} + \frac{568}{197} = \frac{157370}{40779}$, from which we obtain the quotients 3, 1, 6, 10. In the same way, if the work had been carried on to the partial quotient 10, we would have found for the approximate root $x = \frac{12592}{63889} + \frac{250158}{47879}$, which being added, and developed into a continued fraction, gives the quotients 5, 2, 2, 1, 2, 2, 1, 18, 1, 1, 3, and thus the converging fractions may be continued till their terms consist of 11 or 12 figures.

1. Let the equation be $x^3 + 11x^2 - 102x + 181 = 0$.

Quotients, 4, 3, 1, 2, 4, 20, 2, 3, 1, &c.

Conv. fractions, $\frac{1}{0}, \frac{3}{1}, \frac{13}{4}, \frac{16}{5}, \frac{45}{14}, \frac{196}{61}, \frac{3965}{1234}, \frac{8126}{2529}, \frac{28343}{8821}$, &c.

2. Let the equation be $x^4 - x^3 - 3x^2 + 2x + 1 = 0$.

Here 1 is a root; hence, dividing by $x - 1 = 0$, we get the equation $x^3 - 3x - 1 = 0$, in which the partial quotients are found to be 1, 1, 7, 3, 2, 3, 1, 1, 6, 11, and the last transformation gives $z = \frac{1313}{4309} + \frac{21864}{1907} = 11\frac{6325974}{8217263}$, and the development of this fraction gives, after 11, the quotients 1, 3, 2, 1, 9, 1, 2, 5, &c. whence the converging fractions are easily found to be

$$\frac{1}{1}, \frac{2}{1}, \frac{15}{8}, \frac{47}{25}, \frac{109}{58}, \frac{374}{199}, \frac{483}{251}, \frac{857}{456}, \frac{5625}{2993}, \text{ \&c.}$$

3. Let the equation be $x^3 - 63x + 189 = 0$.

Quotients, 4, 14, 6, 1, 4, 2, 1, &c.

Conv. fractions, $\frac{1}{0}, \frac{4}{1}, \frac{57}{14}, \frac{346}{85}, \frac{403}{99}, \frac{1958}{481}, \frac{4319}{1061}, \frac{6277}{1542}$, &c.

OF INDETERMINATE EQUATIONS.

When more quantities are sought than the number of conditions or equations given, the problem is indeterminate or unlimited, but the answers in integers are often limited to a certain number.

I. When the equation is simple, after it is properly reduced, it will assume one of the following forms, viz. $ax = by + c$, $bxy = c - ax$, $ax + cxy = d + by$, where x and y are unknown, and the rest known quantities. The first form $ax - by = c$ is unlimited in the number of answers, for $x = \frac{c + by}{a}$, where y may be any value whatever. Hence x and y admit of an infinite number of values which will satisfy the conditions of the equation, and such is always the case when one of the unknown quantities is entirely dependent upon the value of another unknown quantity. When, however, the equation is of the form $ax + by = c$, the number of answers in integers is limited, for then $x = \frac{c - by}{a}$, and, conse-

quently, in order that the values of x and y shall be integral, the equation is limited to finding all the integral values of y, which will make $\frac{c - by}{a}$ an integer.

In the equation $ax - by = c$, $x = \frac{by + c}{a}$, is an integer, therefore $by + c$ is divisible by a; take such multiples of $by+c$, and of ay, as shall make their difference of the form $y \pm d$ where the coefficient of y is unity, then make $y \pm d = a$ and the least value of y will be found; the other values of y are found by adding a continually, and in like manner the values of x increase continually by b.

To find a number which being divided by 17 shall leave a remainder of 7, and being divided by 26 shall leave 13.

Let x and y be the quotients, then the numbers are $17x + 7 = 26y + 13$; whence $x = \frac{26y + 13 - 7}{17} = \frac{26y + 6}{17}$, an integer; now taking twice $26y + 6$, and thrice $17y$, we have for their difference $\frac{52y + 12 - 51y}{17} = \frac{y + 12}{17}$; then $y + 12 = 17$, or $y = 17 - 12 = 5$, the least value of y; the other values are $5 + 17 = 22$, $22 + 17 = 39$, &c.; and $x = \frac{26y + 6}{17} = 8$, its least value, and the others are $8 + 26 = 34$, $34 + 26 = 60$, &c.

Given $19x - 14y = 11$, whence $x = \frac{11 + 14y}{19}$ and $\frac{44 + 56y - 57y}{19} = \frac{44 - y}{19}$, or $1 + \frac{25 - y}{19}$, a whole number; whence $\frac{25 - y}{19}$ is also a whole number, or $25 - y = 19 \therefore 25 = 19 - y$, or $25 - 19 = 6 = y$, and $x = \frac{11 + 14y}{19} = \frac{14 + 84}{19} = \frac{95}{19} = 5$, the other values of x and y are consequently,

$$x = 5,\ 19,\ 33,\ 47,\ 61,\ \text{\&c.}$$
$$y = 6,\ 25,\ 44,\ 63,\ 82,\ \text{\&c.}$$

Given $21x+17y = 2000$, or $x = \frac{2000 - 17y}{21} = 95 + \frac{5 - 17y}{21}$, a whole number; whence $\frac{5 - 17y}{21}$ is a whole number also, consequently $\frac{25 - 85y + 84y}{21} = \frac{25 - y}{21} = 1$, or $25 - y = 21$, $\therefore 25 = y + 21$, that is $25 - 21 = y$, or $y = 4$. Hence $x = \frac{2000 - 17y}{21} = 92$, its greatest value, all the integral values of x and y are therefore

$$x = 92,\ 75,\ 58,\ 41,\ 24,\ \ 7$$
$$y = 4,\ 25,\ 46,\ 67,\ 88,\ 109.$$

In how many ways is it possible to pay £40 in guineas and crowns only?

Let $x =$ the number of guineas, and $y =$ the number of crowns, then $21x + 5y = 800$, or $x = \frac{800 - 5y}{21} = 38 + \frac{2 - 5y}{21}$, a whole number, rejecting 38, we have $\frac{2 - 5y}{21}$, also a whole number, and $\frac{34 - 85y + 84y}{21} = \frac{34 - y}{21} = 1$, $\therefore 34 - y = 21$, or $34 = y + 21$; hence $34 - 21 = 13 =$ the least value of y, and $x = \frac{800 - 5y}{21} = \frac{800 - 65}{21} = \frac{735}{21} = 35 =$ the greatest value of x; hence, all the positive values of x and y are,

$$x = 35,\ 30,\ 25,\ 20,\ 15,\ \ 10,\ \ 5$$
$$y = 13,\ 34,\ 55,\ 76,\ 97,\ 118,\ 139$$

To compound 100 gallons of spirits, worth 72d., by mixing some at 56d., some at 60d., and some at 80d. per gallon.

Here $x + y + z = 100$, or $20x + 20y + 20z = 2000$.
$56x + 60y + 80z = 7200$, or $14x + 15y + 20z = 1800$.

Whence $6x + 5y = 200$,

or $x = \frac{200 - 5y}{6} = 33 + \frac{2 - 5y}{6}$, an integer, therefore rejecting 33, we have $\frac{2 - 5y + 6y}{6} = \frac{y + 2}{6} = 1$, or $y + 2 = 6 \therefore y$

$= 6 - 2 = 4$, its least value, and $x = 33 + \frac{2 - 5y}{6} = 33 - 3 = 30$, its greatest value, and $z = 100 - 4 - 30 = 66$, its greatest value. The number of solutions is six, namely,

$$\begin{aligned} y &= 4, 10, 16, 22, 28, 34. \\ x &= 30, 25, 20, 15, 10, 5. \\ z &= 66, 65, 64, 63, 62, 61. \end{aligned}$$

When there are three, or more unknown quantities, and only one equation, as

$$ax + by + cz = d.$$

Where c is the greatest coefficient, and where x and y cannot be less than unity, then the value of z cannot be greater than $\frac{d - a - b}{c}$. Having found this limit, we must ascertain the different values of the others, by substituting for the former, separately, from 1 up to the limit found.

Given $3x + 5y + 7z = 100$, to find all possible values of x, y, and z, in whole numbers.

Here, from the nature of the question, each of the least integral values of x and y are 1, whence

$$z = \frac{100 - 5 - 3}{7} = \frac{92}{7} = 13\tfrac{1}{7}.$$

$\therefore$ z cannot be greater than 13, and $x = \frac{100 - 5y - 7z}{3} = 33 - y - 2z + \frac{2y - z + 1}{3}$; therefore $\frac{2y + z - 1}{3}$ is an integer, and $\frac{3y - 2y - z + 1}{3} = \frac{y - z + 1}{3} = 1$, or $y - z + 1 = 3$; hence $y = 3 - 1 + z = 3$ (when $z = 1$). And since $x = \frac{100 - 5y - 7z}{3}$, whence $x = \frac{100 - 15 - 7}{3} = \frac{78}{3} = 26$, and by adding 3, the coefficient of x, continually to this value of y, and subtracting 5, the coefficient of y, from this value of x, we obtain all the possible values of x and y when $z = 1$, thus:

$$z = 1\begin{cases} y = \ \ 3, \ \ 6, \ \ 9, 12, 15, 18 \\ x = 26, 21, 16, 11, \ \ 6, \ \ 1 \end{cases}$$

$$z = 2\begin{cases} y = \ \ 1, \ \ 4, \ \ 7, 10, 13, 16 \\ x = 27, 22, 17, 12, \ \ 7, \ \ 2 \end{cases}$$

$$z = 3\begin{cases} y = \ \ 2, \&c. \\ x = 23, \&c. \end{cases} z = 4\begin{cases} y = \ \ 3, \&c. \\ x = 19, \&c. \end{cases} z = 5\begin{cases} y = \ \ 1, \&c. \\ x = 20, \&c. \end{cases}$$

$$z = 6\begin{cases} y = \ \ 2, \&c. \\ x = 16, \&c. \end{cases} z = 7\begin{cases} y = \ \ 3, \&c. \\ x = 12, \&c. \end{cases} z = 8\begin{cases} y = \ \ 1, \&c. \\ x = 13, \&c. \end{cases}$$

$$z = 9\begin{cases} y = 2, \&c. \\ x = 9, \&c. \end{cases} z = 10\begin{cases} y = 3, \\ x = 5, \end{cases} z = 11\begin{cases} y = 1, \&c. \\ x = 6, \&c. \end{cases}$$

$$z = 12\begin{cases} y = 2 \\ x = 2 \end{cases}$$

Whence all the possible solutions in integers are 41.

To find all the possible values of x, y, and z, in the equation $3x + 2y + 5z = 100$.

$$z = \frac{100 - 3 - 2}{5} = \frac{95}{5} = 19;$$

consequently z cannot be greater than 19. Now, proceeding as before, $x = \dfrac{100 - 2y - 5z}{3} = 33 - z + \dfrac{1 - 2y + 2z}{3}$, and $\dfrac{1 - 2y + 2z}{3}$, is an integer; $\therefore \dfrac{3y - (2y + 2z + 1)}{3} = \dfrac{y - 2z - 1}{3}$ $= 1$, or $y - 2z - 1 = 3$; $\therefore y = 3 + 2z - 1 = 4$, when $z = 1$, and $x = \dfrac{100 - 2y - 5z}{3} = \dfrac{100 - 8 - 5}{3} = \dfrac{87}{3} = 29$.

Now, by adding continually the coefficient of x to this value of y, and subtracting the coefficient of y from this value of x, we obtain all their values when $z = 1$, thus:

$$\begin{aligned} y &= \ \ 4, \ \ 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46, \\ x &= 29, 27, 25, 23, 21, 19, 17, 15, 13, 11, \ \ 9, \ \ 7, \ \ 5, \ \ 3, \ \ 1, \end{aligned}$$

$$z = 2\begin{cases} y = \ \ 6, \&c. \\ x = 26, \&c. \end{cases} z = 3\begin{cases} y = \ \ 8, \&c. \\ z = 23, \&c. \end{cases}$$

and by taking $z = 5, 6,$ &c. up to 19, we find that the number of solutions in integers is 123.

1. Given $17x + 19y + 21z = 400$, to find all the integral values of x, y, and z.

$$\text{Ans. } \begin{array}{rrrrrrrrrrr} z = & 1, & 2, & 3, & 4, & 5, & 6, & 11, & 12, & 13, & 14 \\ y = & 11, & 9, & 7, & 5, & 3, & 1, & 8, & 6, & 4, & 2 \\ x = & 10, & 11, & 12, & 13, & 14, & 15, & 1, & 2, & 3, & 4 \end{array}$$

2. Given $7x - 12y = 19$, to find the least integral values of x and y. Ans. $x = 13$, $y = 6$.

3. Given $27x + 16y = 1600$, to find the least integral values of x and y. Ans. $x = 48$ and $y = 19$.

4. Given $5x + 7y + 11z = 224$, to find all the possible integral values of x, y, and z. Ans. Least value of z is 1.

When $y = 4, 9, 14, 19, 24, 29$.
Then $x = 37, 30, 23, 16, 9, 2$. The number of integral answers being 59.

5. A person bought as many ducks and geese together, as cost him 28s.; for the geese he paid 4s. 4d. a-piece, and for the ducks 2s. 6d. a-pair. How many had he of each?

Ans. 3 geese, and 6 ducks.

6. Forty-one persons, consisting of men, women, and children, spent among them 40s., of which sum each man paid 4s., each woman 3s., and each child 4d. How many were there of each? Ans. 5 men, 3 women, and 33 children.

7. A butcher buys oxen, calves, and sheep, to the number of 100, for £100; the oxen cost him £3, 10s. a-piece, the calves £1, 6s. 8d. each, and the sheep 10s. a-piece. How many had he of each? Ans. 5, 10, or 15 oxen.
42, 24, or 6 calves.
53, 66, or 79 sheep.

8. What number divided by 11, leaves 3 for a remainder, and when divided by 19, leaves 5 for a remainder?

Ans. 4128.

9. A farmer bought some horses, and oxen, and paid £31 for each horse, and £20 for each ox, and he finds that the whole price of the oxen is £7 more than that of the horses. How many had he of each?

Ans. Number of oxen, 5, 36, 67, 98, 129, 160, &c.
Number of horses, 3, 23, 43, 63, 83, 103, &c.

10. A farmer laid out on horses and oxen the sum of £1770, and he paid £31 for each horse, and £21 for each ox. What number had he of each?

Ans. Number of horses, 9, 30, or 51.
Number of oxen, 71, 40, or 9.

11. Two women had together 100 eggs; one said to the

other, when I count mine by eight at a time, there remain 7 over; the second replied, when I count mine by 10 at a time, the remainder is also 7. How many eggs had each?

Ans. The first had 63, or 23 eggs.
The second had 37, or 77 eggs.

12. A goldsmith has three kinds of silver; the first is 7 oz. fine, the second $5\frac{1}{2}$ oz., and the third $4\frac{1}{2}$ oz. fine, per lb.; and he wishes to form a mixture of 30 lbs. so that the fineness may be 6 oz. How much of each sort must he take?

Ans. Of the first sort, 10, 12, 14, 16, or 18 lbs.
Of the second sort, 20, 15, 10, 5, or 0 lbs.
Of the third sort, 0, 3, 6, 9, or 12 lbs.

RESOLUTION OF INDETERMINATE EQUATIONS BY MEANS OF CONTINUED FRACTIONS.

CASE I. When the equation is of the form $ax - by \pm c$, and a and b are prime to each other.

Develop the fraction $\frac{b}{a}$ into a continued fraction, and find the converging fractions; let the fraction immediately preceding $\frac{b}{a}$ be denoted by $\frac{p}{q}$, then will $x = \pm pc$, and $y = \pm qc$.

If the fraction taken be in an odd place, the upper sign is used, otherwise the under sign.

Given $256x - 87y = 50$ to find the integral values of x and y.

Here $a = 256$, $b = 87$, and $c = 50$; hence the fraction to be developed $\frac{87}{256}$ gives for quotients

0, 2, 1, 16, 2, 2.

$\frac{1}{0}, \frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{17}{50}, \frac{35}{103}, \frac{87}{256},$ converging fractions, and that to be taken is $\frac{35}{103}$. Here $p = 35$, and $q = 103$; hence $x = pc = 35 \times 50 = -1750$, and $y = qc = 103 \times 50 = -5150$.

In order to find all the values of x and y, make $x = 87z - 1750$, and $y = 256z - 5150$, and as z may be any num-

ber positive or negative, hence the number of solutions is infinite. If we take $z = 21$ we obtain $x = 87 \times 21 - 1750 = 77$, and $y = 256 \times 21 - 5150 = 226$, which are the least positive values of x and y, and if we continually increase or diminish these values, the former by 87, and the latter by 256, we will obtain an indefinite number of values of x and y.

The indeterminate quantity z may always be taken, so that x shall not exceed $\pm \frac{1}{2}b$, or so that y shall not exceed $\pm \frac{1}{2}a$;

for as $35 \times 50 > 87$ take $z = \frac{35 \times 50}{87} = 20$, the nearest integer; then $x = -10$, and $y = -30$, both less than $\frac{1}{2}$ of 87.

Given $7x - 5y = 8$, to find the values of x and y. Here the fraction $\frac{5}{7}$ gives for quotients

0, 1, 2, 2.

$\frac{1}{0}, \frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{5}{7}$, converging fractions. Here $p = 2$, $q = 3$, and $c = 8$; hence $x = pc = 2 \times 8 = -16$, and $y = 3 \times 8 = -24$.

To find $x < \frac{5}{2}$ take $z = \frac{2 \times 8}{5} = 3$; then $x = 15 - 16 = -1$, and $y = 21 - 24 = -3$. To find $y < \frac{7}{2}$, take $z = \frac{3 \times 8}{7} = 3$; then $y = 21 - 24 = -3$ the same as before. The least positive value of x is 4, and that of y is also 4, and all the other values are found by continually augmenting or diminishing the value of x by 5, and that of y by 7.

Given $89x - 71y = -10$, to find the values of x and y. Here the fraction $\frac{71}{89}$ gives for quotients

0, 1, 3, 1, 17.

$\frac{1}{0}, \frac{0}{1}, \frac{1}{1}, \frac{3}{4}, \frac{4}{5}, \frac{71}{89}$, converging fractions. Here $p = 4$, $q = 5$, and $c = -10$; hence $x = 4 \times -10 = -40$, and $y = 5 \times -10 = -50$, or taking $z = 1$, $x = 71z - 40 = 31$, and y

$= 89z - 50 = 39$, the least positive values, whence the others may be found, by continually augmenting or diminishing the value of x, by 71, and that of y, by 89.

Given $450x - 377y = 250$, to find the values of x and y.

Here the fraction $\frac{377}{450}$, gives for quotients,

0, 1, 5, 6, 12.

$\frac{1}{0}, \frac{0}{1}, \frac{1}{1}, \frac{5}{6}, \frac{31}{37}, \frac{377}{450}$, converging fractions; where $p = 31$, $q = 37$, and $c = 250$; hence $x = 31 \times 250$, and $y = 37 \times 250$; or if $z = 21$, then $x = 377z - 31 \times 250 = 167$, and $y = 450z - 37 \times 250 = 200$, the smallest positive values.

Given $1235x - 987y = -651$, to find the values of x and y. The fraction $\frac{987}{1235}$, gives for quotients,

0, 1, 3, 1, 48, 1, 1, 2.

$\frac{1}{0}, \frac{0}{1}, \frac{1}{1}, \frac{3}{4}, \frac{4}{5}, \frac{195}{244}, \frac{199}{249}, \frac{394}{493}, \frac{987}{1235}$, converging fractions. Here $p = 394$, $q = 493$, and $c = -651$; hence, if z be taken $= 260$, we have $x = 987z - 394 \times -651 = 126$, and $y = 1235z - 493 \times -651 = 157$, the least positive values of x and y.

1. Given $19x - 14y = -11$, to find the least positive values of x and y. Ans. $x = 9$, and $y = 13$.

2. Given $3x - 8y = -16$, to find the least positive values of x and y. Ans. $x = 0$, and $y = 2$.

3. Given $24x - 13y = +16$, to find the least positive values of x and y. Ans. $x = 5$, and $y = 8$.

4. Given $31x - 41y = +52$, to find the least positive values of x and y. Ans. $x = 3$, and $y = 1$.

5. Given $27x - 16y = +7$, to find the least positive values of x and y. Ans. $x = 5$, and $y = 8$.

6. Given $30x - 31y = -34$, to find the least positive values of x and y. Ans. $x = 3$, and $y = 4$.

7. Given $57x - 62y = +27$, to find the least positive values of x and y. Ans. $x = 7$, and $y = 6$.

8. Given $3x - 5y = -3$ to find the least positive values of x and y. Ans. $x = 4$, and $y = 3$.

9. Given $16x - 11y = -6$, to find the least positive values of x and y. Ans. $x = 1$, and $y = 2$.

10. Given $36x - 35y = -43$, to find the least positive values of x and y. Ans. $x = 27$, and $y = 29$.

11. Given $25x - 21y = -36$, to find the least positive values of x and y. Ans. $x = 12$, and $y = 16$.

12. Given $36x - 41y = +143$, to find the least positive values of x and y. Ans. $x = 37$, and $y = 29$.

CASE II. When the equation is of the form $ax + by = c$, it may be solved in the same way as the preceding; and if any integral number z can be taken, so that

$$za > cq, \text{ and } zb < cp,$$

we shall have, for the general values of the unknown quantities,

$$x = cp - zb, \text{ and } y = za - cq.$$

The question will admit of as many solutions as there are different values of z, which will answer these conditions; but if no such value can be assigned to z as will answer these conditions, then the question is impossible.

NOTE. The number of solutions which questions of this kind admit of is always equal to the difference of the integral parts of $\frac{cp}{b} \backsim \frac{cq}{a}$, except when the greater of these is an integer, in which case they will be one less. When the difference between these quantities is less than unity the question is impossible.

Given $3x + 5y = 26$, to find all the values of x and y.

The fraction $\frac{5}{2}$ gives 1, 1, 2, for quotients;

And $\frac{1}{0}, \frac{1}{1}, \frac{2}{1}, \frac{5}{3}$, converging fractions.

Now, since $p = 4$, $q = 3$, and $c = 26$, the general values will be $x = 52 - 5z$, and $y = 3z - 26$, where, in order that $3z - 26$ may be positive, we must take z greater than $\frac{26}{3} = 8\frac{2}{3}$. Hence, taking $z = 9, 10$, &c., we obtain $x = 7, 2$.

$y = 1, 4.$

Given $13x + 17y = 1600$, to find the values of x and y.

The fraction $\frac{17}{3}$ gives 1, 3, 4, for quotients;

And $\frac{1}{0}, \frac{1}{1}, \frac{4}{3}, \frac{17}{13}$, converging fractions.

Now, since $p = 4$, $q = 3$, and $c = 1600$, the general values will be $x = 6400 - 17z$, and $y = 13z - 4800$; where, in order that $13z - 4800$ may be positive, we must take z greater than $\frac{4800}{13} = 369\frac{3}{13}$.

Hence, taking $z = 370, 371, 372$, &c.,
we get $x = 110, 93, 76, 59, 42, 25, 8.$
$y = 10, 23, 36, 49, 62, 75, 88.$

1. Given $11x + 5y = 1031$, to find the values of x and y.
Ans. $x = 91, 86, 81, 76, 71$, &c.
$y = 6, 17, 28, 39, 50$, &c.

2. Given $27x + 16y = 1600$, to find all the possible values of x and y in integers. Ans. $x = 48, 32, 16, 0.$
$y = 19, 46, 73, 100.$

3. Given $12x + 7y = 340$, to find all the values of x and y.
Ans. $x = 26, 19, 12, 5.$
$y = 4, 16, 28, 40.$

4. Given $5x + 27y = 2000$, to find all the values of x and y.
Ans. $x = 400, 373, 346, 319$, &c.
$y = 0, 5, 10, 15$, &c.

5. Given $21x + 17y = 2000$, to find all the values of x and y in integers. Ans. $x = 7, 24, 41, 58, 75, 92.$
$y = 109, 88, 67, 46, 25, 4.$

6. Given $13x + 14y = 200$, to find all the values of x and y. Ans. $x = 10$, and $y = 5$, the only values.

7. Given $140x + 17y = 2000$, to find all the values of x and y. Ans. $x = 7$, and $y = 60$, the only values.

8. Given $42x + 16y = 396$, to find all the values of x and y. Ans. $x = 6$, and $y = 9$, the only values.

9. Given $9x + 13y = 2000$, to find how many answers the equation admits of. Ans. 17 different answers.

10. Given $24x + 17y = 3778$, to find all the values of x and y. Ans. $x = 156, 139, 122, 105, 88$, &c.
$y = 2, 26, 50, 74, 98$, &c.

11. Given $15x + 37y = 7874$, to find how many answers the equation admits of in integers.

Ans. 15 different answers.

12. Given $7x + 9y + 23z = 9999$, to find how many answers the equation admits of in integers.

Ans. 34365 different answers.

SOLUTION OF INDETERMINATE EQUATIONS OF THE SECOND DEGREE.

The general form of indeterminate equations of the second degree is

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

in which x and y are indeterminate, and a, b, c, d, e, f, are known integral numbers, either positive or negative.

CASE I. When $a = 0$, and the equation is

$$bxy + cx^2 + dy + ex + f = 0.$$

Here $y = -\frac{cx^2 + ex + f}{bx + d}$. Multiplying both sides by b^2, and dividing the numerator by the denominator till x disappears in the remainder, we have

$b^2y = -bcx + (cd - be) - \frac{b^2f - bde + cd^2}{bx + d}$, where, if we put $b^2f - bde + cd^2 = r$, it is obvious, since x and y are whole numbers, that $\frac{r}{bx + d}$ must also be a whole number, or that r is divisible by $bx + d$. Let s be one of the divisors of r, then $bx + d = s$, and $x = \frac{s - d}{b}$. In order, therefore, to find x, we must retain only those factors of r, in which $s - d$ is divisible by b.

Let $xy + x^2 = 5x + 6y + 18$, to find the values of x and y.

Here $y = -\frac{x^2 + 5x + 18}{x - 6} = -x - 1 + \frac{12}{x - 6}$, hence $x - 6$ must be a divisor of 12, and the divisors of 12 are 1, 2, 3, 4, 6, and 12. Now, if

$x - 6 = 1$, $x = 7$, and $y = -7 - 1 + \frac{12}{1} = +4$; and if

$x - 6 = 2$, $x = 8$, and $y = -8 - 1 + \frac{12}{2} = -3$.

1. Let $x^2 + xy = 5x + 3y + 30$, to find the values of x and y. Ans. When $x = 4$, $y = 34$.

2. Let $x^2 + xy = 12x + 2y + 22$, to find the values of x and y. Ans. When $x = 9$, $y = 7$.

3. Let $x^2 + xy - 4x + 2y - 30 = 0$, to find the values of x and y. Ans. When $x = 4$, $y = 5$.

4. Let $x^2 - xy - 3x - 4y + 35 = 0$, to find the values of x and y. Ans. When $x = 3$, $y = 5$.

5. Let $x^2 + xy = 6x - 3y + 59$, to find the values of x and y. Ans. When $x = 5$, $y = 8$.

6. Let $x^2 + xy = 29 + 3y + 2x$, to find the values of x and y. Ans. When $x = 4$, $y = 21$.

Case II. The general equation may be written in this form,

$$y^2 + \frac{bx + d}{a} y = - \frac{cx^2 + ex + f}{a},$$

and this quadratic, when resolved in the usual way, gives

$$y + \frac{bx + d}{2a} = \sqrt{\left\{\frac{(bx + d)^2}{4a^2} - \frac{cx^2 + ex + f}{a}\right\}};$$

or $2ay + bx + d = \sqrt{\{(bx + d)^2 - 4a\,(cx^2 + ex + f)\}}$; and putting $2ay + bx + d = z$, $b^2 - 4ac = m$, $2bd - 4ae = n$, and $d^2 - 4af = p$, we have

$$z = \sqrt{(mx^2 + nx + p)};$$

whence the problem is reduced to that of making $\sqrt{(mx^2 + nx + p)}$ rational, where m, n, and p are given integers.

This branch of algebra is usually called the Diophantine Analysis, the principal methods of which are contained in the following problems:

Problem I.

To find such values of x as will make $\sqrt{(mx^2 + nx + p)}$ rational, or $mx^2 + nx + p =$ a square.

There are six cases of this problem which admit of a direct solution.

Case I. When $m = 0$, or $nx + p$, is a square.

Let $nx + p = z^2$, then $x = \frac{z^2 - p}{n}$, where z may be taken at pleasure.

Let $z = \sqrt{(7x+5)}$, to find the value of x.

Here $n = 7$, and $p = 5$; $\therefore x = \frac{z^2 - 5}{7}$, where, if we take $z = 3$, $x = \frac{9-5}{7} = \frac{4}{7}$.

1. Find such a value of x as will make $25x - 7$ a square.

Ans. $x = \frac{8}{25}$, when $z = 1$.

2. Find such a value of x as will make $\sqrt{(x + 14)}$ rational. Ans. $x = 2$, when $z = 4$.

3. Find such a value of x as will make $\sqrt{(x - 20)}$ rational. Ans. $x = 21$, when $z = 1$.

4. Find such a value of x as will make $50x - 70$ a square.

Ans. $x = \frac{17}{5}$, when $z = 10$.

5. Find such a value of x as will make $\frac{1}{4}x + 7$ a square.

Ans. $x = 4$, when $z = 3$.

6. Find such a value of x as will make $\sqrt{\left(\frac{3}{4}x - \frac{2}{7}\right)}$ rational. Ans. $x = \frac{12}{7}$, when $z = 1$.

Case II. When $p = 0$, or $mx^2 + nx = (mx + n)x$, is a square.

Let $mx^2 + nx = r^2x^2$, as it must be a square; then $mx + n = r^2x$, and $x = \frac{n}{r^2 - m}$, where r may be taken at pleasure.

Let $z = \sqrt{(7x^2 + 3x)}$, to find the values of z and x.

Here $m = 7$, $n = 3$, and $x = \frac{n}{r^2 - m} = \frac{3}{r^2 - 7}$, and if r

is taken $= 3$, $x = \frac{3}{2}$, and $z = \sqrt{\{7 \times \left(\frac{3}{2}\right)^2 + \left(3 \times \frac{3}{2}\right)\}} = \sqrt{\left(\frac{63}{4} + \frac{18}{4}\right)} = \sqrt{\frac{81}{4}} = \frac{9}{2}$.

1. Let $z = \sqrt{(3x^2 + 4x)}$, to find the values of x and z.

Ans. When $r = 2$, $x = 4$, and $z = 8$.

2. Let $z = \sqrt{(5x - 2x^2)}$, to find the values of x and z.

Ans. When $r = 1$, $x = \frac{5}{3}$, and $z = \frac{5}{3}$.

3. Let $z^2 = 7x - 3x^2$, to find the values of x and z.

Ans. When $r = 1$, $x = \frac{7}{4}$, and $z = \frac{7}{4}$.

4. Let $z^2 = \frac{4}{5}x^2 + \frac{9}{11}x$, to find the values of x and z.

Ans. When $r = 1$, $x = \frac{45}{11}$, and $z = \frac{45}{11}$.

5. To find two numbers such, that the square of the first, with twice the square of the second, shall be equal to 10 times the second.

Ans. When $r = 1$, the first is $\frac{10}{3}$, and the second is $\frac{10}{3}$.

6. To find two numbers such, that the square of the first, *minus* 3 times that of the second, shall be equal to 4 times the second.

Ans. When $r = 1$, the first is $= 8$, and the second is $= 4$.

Case III. When p is a square number $= k^2$, or the equation is of the form

$$z^2 = mx^2 + nx + k^2.$$

Assume $z = ux + k$, and we have

$$z^2 = u^2x^2 + 2kxu + k^2 = mx^2 + nx + k^2;$$

$$\text{or } xu^2 + 2ku = mx + n;$$

whence $x = \frac{2ku - n}{m - u^2}$, where u may be taken at pleasure.

Let $z^2 = 4x^2 + 2x + 16$, to find the values of x and z.

Here $m = 4$, $n = 2$, $k = 4$, and $x = \frac{8u - 2}{4 - u^2}$, where, if $u = 1$, $x = \frac{8 - 2}{4 - 1} = \frac{6}{3} = 2$; and $z^2 = 16 + 4 + 16 = 36$, $\therefore z = 6$.

1. Let $z^2 = 2x^2 - 4x + 16$, to find the values of x and z.

Ans. When $u = 1$, $x = 12$, and $z = 16$.

2. Let $z = \sqrt{(5x^2 + 3x + 36)}$, to find the values of x and z.

Ans. When $u = 1$, $x = \frac{9}{4}$, and $z = \frac{33}{4}$.

3. To find two numbers such, that the square of the first, *minus* 6 times that of the second, shall be equal to 25, *minus* 4 times the second.

Ans. When $u = 1$, the first is $= 7\frac{4}{5}$, and the second is $= 2\frac{4}{5}$.

4. To find two numbers such, that the square of the first, *minus* 3 times that of the second, shall be equal to 36, *minus* 5 times the second.

Ans. When $u = 1$, the first is $= 7\frac{1}{4}$, and the second is $= 8\frac{1}{2}$.

5. To find two numbers such, that the square of the first, with twice that of the second, shall be equal to 3 times the second, *plus* 49.

Ans. When $u = -1$, the first is $= 1\frac{1}{3}$, and the second is $= 5\frac{2}{3}$.

6. To find two numbers such, that the square of the first, with 4 times that of the second, shall be equal to 64.

Ans. When $u = -1$, the first is $= 4\frac{4}{5}$, and the second is $= 3\frac{1}{5}$.

CASE IV. When m is a square number $= k^2$, or the equation is of the form

$$z^2 = k^2x^2 + nx + p.$$

Assume $z = kx + u$, and we have then

$$z^2 = k^2x^2 + 2kxu + u^2 = k^2x^2 + nx + p;$$

$$\text{or } 2kxu + u^2 = nx + p;$$

whence $x = \frac{u^2 - p}{n - 2ku}$, where u may be taken at pleasure.

Let $z^2 = 4x^2 + 7x - 3$, to find the values of x and z.

Here $k = 2$, $n = 7$, and $p = -3$; hence $x = \frac{u^2 + 3}{7 - 4u}$, and

if $u=1$, $x=\frac{1+3}{7-4}=\frac{4}{3}$, and $z=\sqrt{\left\{4\times\left(\frac{4}{3}\right)^2+\left(7\times\frac{4}{3}-3\right)\right\}}$

$$=\sqrt{\left(\frac{64}{9}+\frac{84}{9}-\frac{27}{9}\right)}=\sqrt{\frac{121}{9}}=\frac{11}{3}.$$

1. Let $z^2 = 25x^2 + 3x + 7$, to find the values of x and z.

Ans. When $u = -3$, $x = \frac{2}{33}$, and $z = \frac{89}{33}$.

2. Let $z^2 = 9x^2 + 5x - 11$, to find the values of x and z.

Ans. When $u = -1$, $x = \frac{12}{11}$, and $y = \frac{25}{11}$.

3. Let $z^2 = -7x + 13 + 4x^2$, to find the values of x and z.

Ans. When $u = -4$, $x = \frac{1}{3}$, and $z = 3\frac{1}{3}$.

4. To find two numbers such, that the square of the first, *minus* 4 times the square of the second, is equal to 15, *plus* 3 times the second.

Ans. When $u = -4$, the first $= \frac{74}{19}$, and the second $= \frac{1}{19}$.

5. To find two numbers such, that the square of the first, *plus* 7 times the second, is equal to 3, *plus* 9 times the square of the second.

Ans. When $u = -2$, the first $= \frac{7}{5}$, and the second $= \frac{1}{5}$.

6. To find two numbers such, that 11 times the second, *plus* 16 times its square, is equal to 5, *plus* the square of the first.

Ans. When $u = 1$, the first $= 9$, and the second $= 2$.

CASE V. When $mx^2 + nx + p$ can be resolved into two simple factors, $f+gx$, and $h+kx$, which can always be done when $n^2 - 4pm$ is a square; then

$$z^2 = (f+gx)\times(h+kx).$$

Assume $z = \sqrt{\{(f+gx)(h+kx)\}} = u(f+gx)$; then

$$(f+gx)(h+kx) = u^2(f+gx)^2;$$

whence $x = \frac{fu^2 - h}{k - gu^2}$, and $z = \sqrt{\{(f + gx)\ (h + kx)\}}$;

whence $z = \frac{(fk - gh)u}{k - gu^2}$, where u may be taken at pleasure.

Let $z^2 = 15 + 19x + 6x^2$, to find the value of x and y.

Here $n^2 - 4pm = 19^2 - 4 \times 15 \times 6 = 361 - 360 = 1$, a square, and the two factors are found to be $3 + 2x$, and $5 + 3x$; here $f = 3$, $g = 2$, $h = 5$, and $k = 3$; $\therefore x = \frac{3u^2 - 5}{3 - 2u^2}$;

$$\text{whence, if } u = \frac{5}{4},\ x = \frac{\frac{75}{16} - 5}{3 - \frac{50}{16}} = \frac{5}{2} = 2\tfrac{1}{2},$$

$$\text{and } z = \frac{\{(3 \times 3) - (2 \times 5)\}\frac{5}{4}}{3 - \left(2 \times \frac{25}{16}\right)} = \frac{\frac{5}{4}}{\frac{2}{16}} = 10.$$

1. Let $z^2 = 8x^2 + 14x + 6$, to find the values of x and y.

Ans. If $u = \frac{11}{4}$, $x = 3\frac{4}{7}$, and $z = 12\frac{4}{7}$.

2. Let $z^2 = 2x^2 - 2$, to find the values of x and z.

Ans. When $u = 1$, $x = 3$, and $z = 4$.

3. To find two numbers such, that the square of the first, *minus* 36 times the second, shall be equal to 5 times the square of the second, *plus* 7.

Ans. When $u = 2$, the first $= 68$, and the second $= 27$.

4. To find two numbers such, that the square of the first, *minus* 15, shall be equal to 19 times the second, *plus* 6 times its square.

Ans. When $u = 1$, the first $= 1$, and the second $= -2$.

5. To find two numbers such, that the square of the first, *plus* 8 times the second, shall be equal to the square of the second, *plus* 7.

Ans. When $u = 2$, the first $= 4$, and the second $= 9$.

6. To find two numbers such, that the square of the first, *minus* 2, shall be equal to 3 times the square of the second, *minus* 5 times itself.

Ans. When $u = 1$, the first $= \frac{1}{2}$, and the second $= \frac{1}{2}$.

CASE VI. When mx^2+nx+p can be separated into two parts, one of which is a square, and the other the product of two simple factors, that is, when $\sqrt{(mx^2+nx+p)} = \sqrt{\{(ax+e)^2+(f+gx)(h+kx)\}}$.

Assume $\sqrt{\{(ax+e)^2+(f+gx)(h+kx)\}} = \{(ax+e)+u(f+gx)\}^2$; then $(ax+e)^2+(f+gx)(h+kx) = (ax+e)^2 + 2u(ax+e)(f+gx)+u^2(f+gx)^2$.

$$\text{Or } h+kx = 2u(ax+e)+u^2(f+gx);$$

$$\text{whence } x = \frac{u(2e+fu)-h}{k-u(2a+gu)}.$$

Let $z^2 = 6x^2+13x+10$, to find the values of x and z.

Here one part is $4 = 2^2$, and the other $6x^2+13x+6$, and since $13^2 - 4(6\times 6) = 25$ is a square, we can divide it into two simple factors, which are found to be $2+3x$, and $3+2x$.

Now, since $a = 0$, $e = 2$, $f = 2$, $g = 3$, $h = 3$, and $k = 2$; $\therefore x = \frac{u(4-2u)-3}{2-u(3u)}$; whence, if $u = 1$, $x = \frac{4+2-3}{2-3} = -3$, and $z = 5$.

1. Let $z^2 = 13x^2+15x+7$, to find the values of x and z.
Ans. When $u = 1$, $x = 3$, and $z = 13$.

2. To find two numbers such, that the square of the greater, *plus* 7 times the less, shall be equal to 7 times the square of the less, *minus* 5.
Ans. When $u = 0$, the greater $= 3$, and the less $= 2$.

3. To find two numbers such, that the square of the greater, *minus* 13 times the less, shall be equal to 5 times the square of the less, *plus* 10.
Ans. When $u = 2$, the greater $= 70$, and the less $= 30$.

4. To find two numbers such, that the square of the greater, *minus* 19, shall be equal to 13 times the less, *plus* 7 times its square.
Ans. When $u = 1$, the greater $= \frac{11}{3}$, and the less $= -\frac{2}{3}$.

5. To find two numbers such, that the square of the greater, *minus* 29 times the less, shall be equal to 4 times the square of the less, *plus* 61.
Ans. When $u = \frac{1}{2}$, the greater $= 13\frac{2}{7}$, and the less $= 2\frac{6}{7}$.

6. To find two numbers such, that the square of the greater, *plus* 7 times the less, shall be equal to 16 times the square of the less, *minus* 2.

Ans. When $u = 1$, the greater $= 11$, and the less $= 3$.

The preceding six methods comprise all the cases which can be resolved by any direct rule; but either in these, or in other instances of a different kind, if we can find by trial any simple value of the unknown quantity which satisfies the conditions of the question, then other values may be easily found.

Thus, in the formula $z^2 = mx^2 + nx + p$, let r be such a value of x as will make the expression a square, and assume $mr^2 + nr + p = s^2$, then putting $x = y + r$, we have

$$mx^2 + nx + p = m(y+r)^2 + n(y+r) + p.$$
$$= my^2 + (2mr + n)y + mr^2 + nr + p.$$

And since this is of the same form as that under Case III., we may thence deduce the value of y, and consequently that of $x = y + r$.

Let $z^2 = 7x^2 + 2$, to find the value of x and z.

Here it is manifest, that if $x = 1$, the expression is a square; let, therefore, $x = y + 1$, and by substitution we have $z^2 = 7x^2 + 2 = 7y^2 + 14y + 9$.

Where $m = 7$, $n = 14$, and $k = 3$, whence, by Case III.,

$y = \frac{6u - 14}{7 - u^2}$, and if $u = 1$, $y = \frac{6 - 14}{7 - 1} = -\frac{4}{3}$, and as $x = y + 1$, $x = 1 - \frac{4}{3} = -\frac{1}{3}$, and $z = \frac{5}{3}$; or if $u = \frac{8}{3}$,

$y = \frac{16 - 14}{7 - \frac{64}{9}} = -\frac{\frac{2}{1}}{\frac{1}{9}} = -18$, and $x = 1 + y = 1 - 18$

$= -17$; whence $z = 55$.

1. Let $z^2 = 5x^2 + 19$, to find the values of x and z.

When $x = 3$, the expression is a square; whence, by putting $x = y + 3$, and proceeding as before, we find

When $u = 2$, $y = 2$, $x = 5$, and $z = 12$.

When $u = 3$, $y = -\frac{9}{2}$, $x = -\frac{3}{2}$, and $z = \frac{11}{2}$.

2. To find two numbers such, that the square of the greater, *minus* 4 times the less, shall be equal to 5 times the square of the less, *plus* 7.

Ans. When $x = 1$, the conditions are satisfied; proceeding therefore as before, when $u = 2$, $y = 2$, $x = 3$, and $z = 8$.

3. To find two numbers such, that the square of the greater, *minus* 4 times the square of the less, shall be equal to 12 times the less, *minus* 31.

Ans. $x = 4$ satisfies the conditions of the question; whence if $u = 3$, $y = -2$, $x = 2$, and $z = 3$.

4. To find two numbers such, that the square of the greater, *plus* 3 times the less, shall be equal to 5 times the square of the less, *plus* 2.

Ans. $x = 2$ satisfies the conditions of the question; whence, if $u = 3$, $y = -\frac{7}{4}$, and the less is $= \frac{1}{4}$, and the greater $= 3\frac{1}{16}$.

5. To find two numbers such, that the square of the greater, *plus* 4, is equal to 5 times the square of the less.

Ans. $x = 1$ satisfies the conditions of the question; whence, if $u = 2$, the less $= -5$, and the greater $= 11$.

6. To find two numbers such, that the square of the greater, *minus* 8, shall be equal to 12 times the less, *plus* 5 times its square.

Ans. $x = 1$ satisfies the conditions of the question; whence, if $u = 1$, $x = -2$, and $z = 2$.

Problem II.

To find such values of x as will make $\sqrt{(ax^3 + bx^2 + cx + d)}$ rational, or $ax^3 + bx^2 + cx + d =$ a square.

There are only two cases of this problem which admit of a direct solution.

Case I. When the two last terms of the formula are wanting, and it is of the form

$$\sqrt{(ax^3 + bx^2)}, \text{ or } ax^3 + bx^2 = \text{a square.}$$

Assume $\surd(ax^3 + bx^2) = nx$, or $ax^3 + bx^2 = n^2x^2$, and divide each side by x^2; then $ax+b = n^2$, whence $x = \frac{n^2-b}{a}$.

To find a number such, that if 5 times its cube be added to 3 times its square, the sum shall be a square.

Assume the number $5x^3+3x^2 = n^2x^2$, then $5x+3 = n^2$, or $x = \frac{n^2-3}{5}$, where, if we take $n = 3$, $x = 1\frac{1}{5}$, and if $n =$ 4, 5, 6, 7, 8, &c. we obtain respectively $x = 2\frac{3}{5}, 4\frac{2}{5}, 6\frac{3}{5}, 9\frac{1}{5}$, $12\frac{1}{5}$, &c.

1. To find a number such, that 7 times its cube *plus* 5 times its square, is a square. Ans. If $n = 4$, $x = 1\frac{4}{7}$.

2. To find a number such, that 4 times its cube *minus* 7 times its square, shall make a square.
Ans. If $n = 1$, $x = 2$.

3. To find a number such, that its cube *minus* its square, shall be a square.
Ans. If $n = 1$, $x = 2$; if $n = 2$, $x = 5$, &c.

4. To find a number such, that its cube *plus* its square, shall be a square.
Ans. If $n = 2$, $x = 3$; and if $n = 3$, $x = 8$, &c.

CASE II. When the last term of the formula is a square, and it is of the form

$$\surd(ax^3+bx^2+cx+d^2).$$

Assume $\surd(ax^3+bx^2+cx+d^2) = d + \frac{c}{2d}x$.

Then $\quad ax^3+bx^2+cx+d^2 = d^2+cx+\frac{c^2}{4d^2}x^2$,

Or $\quad ax^3+bx^2 = \frac{c^2}{4d^2}x^2$; dividing this by x^2,

we have $ax+b = \frac{c^2}{4d^2}$, and $x = \frac{c^2-4bd^2}{4ad^2}$.

Let $3x^3 - 5x^2+6x+4 =$ a square, to find the value of x.

Here $a = 3$, $b = -5$, $c = 6$, and $d^2 = 4$; hence $x = \frac{36+80}{48} = \frac{116}{48} = \frac{29}{12} = 2\frac{5}{12}$.

1. To find such a value of x as will make $\surd(x^3 - 2x^2 + 2x + 1)$ rational. Ans. $x = 3$.

2. To find such a value of x as will make $\surd(3x^3 + 4x^2 - 5x + 16)$ rational. Ans. $x = -\frac{77}{64}$.

3. To find such a value of x as will make $\surd(2x^3 + 3x^2 + 2x + 25)$ rational. Ans. $x = -\frac{37}{25}$.

4. To find such a value of x as will make $\surd(x^3 - 4x^2 + 3x + 9)$ rational. Ans. $x = 4\frac{1}{4}$.

When it is already known that if $x = n$, the formula, $\surd(ax^3 + bx^2 + cx + d)$ is a square, then other values of x may be determined from this as follows:

Assume $an^3 + bn^2 + cn + d = m^2$, and, putting $x = y + n$, we obtain $an^3 + bn^2 + cn + d = a(y+n)^3 + b(y+n)^2 + c(y+n) + d = ay^3 + (3an + b)y^2 + (3n^2 + 2n + c)y + an^3 + bn^2 + cn + d$, or

$ax^3 + bx^2 + cx + d = ay^3 + (3an + b)y^2 + (3n^2 + 2n + c)y + m^2$, from which, by Case II., the value of y may be found, and thence that of x.

Let $x^3 - 2x^2 + 2x + 1 =$ a square; we found in last case, one value of $x = 3$. Now, putting $x = y + 3$, we obtain $(y+3)^3 - 2(y+3)^2 + 2(y+3) + 1 = x^2 - 2x^2 + 2x + 1$, and the left hand side of this equation, when reduced, becomes $y^3 + 7y^2 + 17y + 16$, the left hand term of which is a square; here $a = 1$, $b = 7$, $c = 17$, and $d^2 = 16$; $\therefore y = \frac{289 - 448}{64} = -\frac{159}{64}$, and $x = 3 - \frac{159}{64} = \frac{33}{64}$.

1. To find such values of x as will make $x^3 + 3$ a square, $x = 1$ being one of the values. Ans. $x = -\frac{23}{16}$.

2. When $x = 3$, the expression $2x^3 - 5$ is a square, to find another value of x which will make it a square.

Ans. $x = -\frac{153}{98}$.

3. When $x = 2$, $3x^3 - 2x^2 + 3x + 3$ is a square, to find another value of x which will make it a square.

Ans. $x = -\frac{213}{100}$.

4. The expression $x^3 - 2x^2 + 3x + 2$ is a square, when $x = 3$, find another value of x which will make it a square.

Ans. $x = -\frac{31}{16}$.

NOTE. If the given formula can be resolved into two factors, one of which is a square; then if the other can be made a square, the whole expression will be so, since the product of two squares is always a square: thus, in the expression $\sqrt{(2x^3 + 10x^2 + 6x - 18)}$, which is the product of $(x+3)^2 \times (2x - 2)$, all which is here required is to find such values of x as will make $2x - 2$ a square, which is readily done by Problem I., Case I.

Few questions in this problem admit of integral answers, many admit of but one answer, and many more are impossible.

PROBLEM III.

To find such values of x as will make

$$\sqrt{(ax^4 + bx^3 + cx^2 + dx + e)} \text{ rational.}$$

There are only three cases of this problem in which direct solutions can be obtained.

CASE I. When the last term only of the formula is a square, or when it is of the form

$$\sqrt{(ax^4 + bx^3 + cx^2 + dx + e^2)}.$$

Assume $ax^4 + bx^3 + cx^2 + dx + e^2 = (rx^2 + sx + e)^2 = r^2x^4 + 2rsx^3 + (s^2 + 2re)x^2 + 2sex + e^2$; whence, putting $2se = d$, or $s = \frac{d}{2e}$, and $s^2 + 2re = c$, or $r = \frac{c - s^2}{2e} = \frac{4ce^2 - d^2}{8e^3}$, the last three terms of both sides destroy each other, and we obtain

$$\begin{aligned} ax^4 + bx^3 &= r^2x^4 + 2rsx^3; \text{ dividing by } x^3, \\ ax + b &= r^2x + 2rs; \\ \text{whence} \quad x &= \frac{2rs - b}{a - r^2}; \end{aligned}$$

and if the values of r and s be substituted in this, we have

$$x = \frac{8e^2\{d(4ce^2 - d^2) - 8be^4\}}{64ae^6 - (4ce - d^2)^2}.$$

This formula, however, fails when b and d are each 0.

Find such a value of x as will make $5x^4 - 4x^3 + 3x^2 - 2x + 1$ a square.

Here $a = 5$, $b = -4$, $c = 3$, $d = -2$, and $e = 1$;

$$\therefore\ x = \frac{8\{-2(12-4)+32\}}{320-(12-4)^2} = \frac{8(-16+32)}{320-64} = \frac{128}{256} = \tfrac{1}{2}.$$

Or since we find $r = 1$, and $s = -1$, we have

$$5x^4 - 4x^3 + 3x^2 - 2x + 1 = (x^2 - x + 1)^2 =$$
$$x^4 - 2x^3 + 3x^2 + 2x + 1;$$
$$\therefore\ 5x^4 - 4x^3 = x^4 - 2x^3,$$
$$\text{or } 5x - 4 = x - 2;$$
consequently $x = \frac{2}{4} = \frac{1}{2}$, as before.

1. Find such a value of x as will make $7x^4 + 3x^3 - 5x^2 + 3x + 4$ a square. Ans. $x = \frac{2665194}{2654336}$.

2. Find such a value of x as will make $2x^4 - 5x^3 + 3x^2 - 2x + 9$ a square. Ans. $x = \frac{3411}{1289}$.

3. Find such a value of x as will make $\sqrt{(-2x^4 + 3x^3 - x^2 - 2x + 4)}$ rational. Ans. $x = \frac{688}{537}$.

4. Find such a value of x as will make $\sqrt{(2x^4 + 8x^3 + 12x^2 + 8x + 1)}$ rational. Ans. $x = 12$.

CASE II. When the first term only of the formula is a square, or when it is of the form

$$\sqrt{(a^2x^4 + bx^3 + cx^2 + dx + e)}.$$

Assume $a^2x^4 + bx^3 + cx^2 + dx + e = (ax^2 + rx + s) = a^2x^4 + 2arx^3 + (r^2 + 2as)x^2 + 2rsx + s^2$; where, putting $2ar = b$, or $r = \frac{b}{2a}$, and $r^2 + 2as = c$, or $s = \frac{c - r^2}{2a} = \frac{4a^2c - b^2}{8a^3}$, the first three terms of both sides destroy each other, and we obtain $dx + e = 2rsx + s^2$;

$$\therefore\ x = \frac{s^2 - e}{d - 2rs}.$$

and, if the values of r and s be substituted in this, we have

$$x = \frac{(4a^2c - b^2)^2 - 64a^6e}{8a^2\{8a^4d - b(4a^2c - b^2)\}}.$$

This formula, like that in the first case, fails when b and d are both 0.

To find a value of x which will make $9x^4 - 4x^3 - 2x^2 + 5x - 3$ a square.

Here $r = -\frac{2}{3}$, and $s = -\frac{11}{27}$; hence

$$9x^4 - 4x^3 - 2x^2 + 5x - 3 = \left(3x^2 - \frac{2}{3}x - \frac{11}{27}\right)^2 =$$

$$9x^4 - 4x^3 - 2x^2 + \frac{44}{81}x + \frac{121}{729};$$

$$\therefore\ 5x - 3 = \frac{44}{81}x + \frac{121}{729},$$

$$\text{or}\quad 3645x - 396x = 2187 + 121,$$

$$\text{and } x = \frac{2308}{3249}.$$

1. To find such a value of x as will make $4x^4 - 2x^3 + 3x^2 + 5x - 7$ a square. Ans. $x = \frac{1913}{1456}$.

2. To find such a value of x as will make $16x^4 + 12x^3 - 8x^2 + 3x - 10$ a square. Ans. $x = \frac{11921}{7008}$.

3. To find such a value of x as will make $\sqrt{(x^4 - 2x^3 + 3x^2 + 4x - 6)}$ rational. Ans. $x = 3$.

4. To find such a value of x as will make $\sqrt{(4x^4 - 2x^3 - x^2 + 3x - 2)}$ rational. Ans. $x = \frac{537}{688}$.

Case III. When both the first and last terms of the formula are squares, or when it is of the form

$$\sqrt{(a^2x^4 + bx^3 + cx^2 + dx + e^2)}.$$

Assume $a^2x^4 + bx^3 + cx^2 + dx + e^2 = (ax^2 + rx + e)^2 = a^2x^4 + 2arx^3 + (r^2 + 2ae)x^2 + 2rex + e^2$; where, putting

$2ar = b$, or $r = \frac{b}{2a}$, $2re = d$, or $r = \frac{d}{2e}$, and $r^2 + 2ae = c$, we have

$$(r^2 + 2ae)\, x^2 + 2rex = cx^2 + dx\,;$$

$$\text{whence } x = \frac{d - 2re}{r^2 + 2ae - c}.$$

Where, substituting the value of r, we obtain

$$x = \frac{4a\,(ad - be)}{b^2 + 4a^2\,(2ae - c)}\,;$$

and since e may be either positive or negative, its second power only being found in the formula, we therefore obtain, when e is negative,

$$x = \frac{4a\,(ad + be)}{b^2 - 4a^2\,(2ae + c)}.$$

Now, since r may be put either equal to $\frac{b}{2a}$, or $\frac{d}{2e}$, and since from the former value of r we obtain two values of x, so from the latter value we also obtain two values; thus, when e is positive,

$$x = \frac{d^2 + 4e^2\,(2ae - c)}{4e\,(be - ad)}\,;$$

$$\text{and also } x = \frac{d^2 - 4e^2\,(2ae + c)}{4e\,(be + ad)}.$$

We may therefore in this case obtain four values of x; but the formulæ fail when b and d are both 0.

This case is evidently included in the two former cases, and therefore other solutions may be obtained by employing the formula in these two cases.

Find such values of x as will make $9x^4 - 3x^3 + 5x^2 + 3x + 9$ a square.

Here $r = -\frac{3}{6} = -\frac{1}{2}$; taking e negative, we have $9x^4 - 3x^3 + 5x^2 + 3x + 9 = (3x^2 - \frac{1}{2}x + 3)^2 = 9x^4 - 3x^3 + \frac{73}{4}x^2 - 3x + 9$;

$$\therefore\ \frac{73}{4}x^2 - 3\,x = 5x^2 + 3x,$$

$$\text{or } 73x^2 - 12x = 20x^2 + 12x\,;$$

$$\text{whence } x = \frac{24}{53}.$$

Again, taking e positive, and $r = \frac{1}{2}$, we have
$9x^4 - 3x^3 + 5x^2 + 3x + 9 = (3x^2 + \frac{1}{2}x + 3)^2 =$
$9x^4 + 3x^3 + \frac{73}{4}x^2 + 3x + 9;$

$$\therefore\ 3x^3 + \frac{73}{4}x^2 = -3x^3 + 5x^2,$$

$$\text{or}\ \ 12x^3 + 73x^2 = -12x^3 + 20x^2;$$

$$\text{whence}\ x = -\frac{53}{24}.$$

1. To find such values of x as will make $\sqrt{(4x^4 + 2x^3 - 4x^2 - 6x + 9)}$ rational. Ans. $x = -\frac{36}{65}$, or $-\frac{5}{6}$.

2. To find such values of x as will make $\sqrt{(x^4 - 2x^3 + 3x^2 + 2x + 4)}$ rational. Ans. $x = 3$, or $-\frac{5}{12}$.

3. To find such values of x as will make $4x^4 - 2x^3 + 7x^2 + 3x + 1$ a square. Ans. $x = -\frac{16}{11}$, or $\frac{21}{16}$.

4. To find such values of x as will make $16x^4 - 6x^3 + 4x^2 + 3x + 4$ a square. Ans. $x = \frac{32}{67}$, or $-\frac{67}{64}$.

When neither the first nor the last term of the formula is a square, we cannot obtain a direct solution; but if by trial we can discover some simple value of the unknown quantity that will satisfy the conditions of the question, we may thence find other values of it, when such are possible.

Suppose we have found by trial that $ax^4 + bx^3 + cx^2 + dx + e$ is a square, when $x = p$, let us then make

$$ap^4 + bp^3 + cp^2 + dp + e = q^2.$$

Assume $x = y + p$, and we have

$$\begin{array}{lcl}
ax^4 & = & ay^4 + 4apy^3 + 6ap^2y^2 + 4ap^3y + ap^4 \\
bx^3 & = & \qquad\qquad\ by^3 + 3bpy^2 + 3bp^2y + bp^3 \\
cx^2 & = & \qquad\qquad\qquad\qquad\ cy^2 + cpy + cp^2 \\
dx & = & \qquad\qquad\qquad\qquad\qquad\quad dy + dp \\
e & = & \qquad\qquad\qquad\qquad\qquad\qquad\quad + e
\end{array}$$

$$\square^{*} = ay^4 + \left.\begin{array}{r}4ap\\+\ b\end{array}\right| y^3 + \left.\begin{array}{r}6ap^2\\+3bp\\+\ c\end{array}\right| y^2 + \left.\begin{array}{r}4ap^3\\+3bp^2\\+\ cp\\+\ d\end{array}\right| y + q^2$$

* The symbol $\square$ denotes a square.

Whence we may find the value of y, by Case I., and then that of x.

To find such a value of x as will make $2x^4 - 1 = \square$.

Here it is obvious that 1 is such a value of x as will make the expression a square.

$\therefore$ Let $x = y + 1$; then

$2x^4 - 1 = 2(y+1)^4 - 1 = 2(y^4 + 4y^3 + 6y^2 + 4y + 1) - 1 = 2y^4 + 8y^3 + 12y^2 + 8y + 1$.

And since the last term of this expression is a square, we have, by Case I.,

$2y^4 + 8y^3 + 12y^2 + 8y + 1 = (-2y^2 + 4y + 1)^2 = 4y^4 - 16y^3 + 12y^2 + 8y + 1$;

$$\therefore\ 4y^4 - 16y^3 = 2y^4 + 8y^3,$$

$$\text{or } 4y - 16 = 2y + 8;$$

$$\text{whence } y = 12, \text{ and } x = 12 + 1 = 13.$$

1. Find such a value of x, as will make $3x^4 + 6 =$ a square.

Ans. $x = 1$ answers the conditions; whence, by Case I., x also $= \frac{12}{11}$.

2. Find such a value of x, as will make $2x^4 + 2x^3 - 2x^2 + x + 7$, a square.

Ans. $x = 2$ answers the conditions; whence, by Case I., x also $= -\frac{1003862}{8048177}$.

3. Find such a value of x, as will make $2x^4 - 3x^2 + 9$, a square.

Ans. $x = 3$ answers the conditions; whence, by Case I., x also $= \frac{40971}{6041}$.

4. Find such a value of x, as will make $3x^4 + 3x^3 - x^2 + 1$, a square.

Ans. $x = 1$ answers the conditions; whence, by Case I., x also $= \frac{945}{10513}$.

PROBLEM IV.

To find such values of x as will make

$ax^3 + bx^2 + cx + d$ a cube.

This problem resolves itself into three cases, namely,

CASE I. When the last term of the formula is a cube, or when it is of the form

$$ax^3 + bx^2 + cx + d^3,$$

assume $ax^3 + bx^2 + cx + d^3 = (rx + d)^3 = r^3x^3 + 3r^2dx^2 + 3rd^2x + d^3$; where $3rd^2 = c$, or $r = \frac{c}{3d^2}$; whence

$$ax^3 + bx^2 = r^3x^3 + 3r^2dx^2,$$

$$\text{or } ax + b = r^3x + 3r^2d;$$

$$\therefore \quad x = \frac{3r^2d - b}{a - r^3}.$$

Or, substituting for r its value $\frac{c}{3d^2}$, as above, we have

$$x = \frac{9d^3(c^2 - 3bd^3)}{27ad^6 - c^3}.$$

Find such a value of x, as will make $2x^3 - 3x^2 + 5x + 1$, a cube.

Here $r = \frac{5}{3}$; $\therefore 2x^3 - 3x^2 + 5x + 1 = \left(\frac{5}{3}x + 1\right)^3 =$

$$\frac{125}{27}x^3 + \frac{25}{3}x^2 + 5x + 1;$$

$$\text{whence } \frac{125}{27}x^3 + \frac{25}{3}x^2 = 2x^3 - 3x^2,$$

or, reducing and dividing by x^2,

$$125x - 54x = -81 - 225;$$

$$\therefore \quad x = -\frac{306}{71}.$$

1. Find such a value of x, as will make $3x^3 + 4x^2 + 7x + 8$, a cube. Ans. $x = -\frac{3384}{4841}$.

2. Find such a value of x, as will make $x^3 - 7x^2 + x - 27$, a cube. Ans. $x = \frac{68769}{9841}$.

3. Find such a value of x, as will make $\sqrt[3]{(2x^3 - 5x^2 + 3x - 1)}$, rational. Ans. $x = 1$.

4. Find such a value of x, as will make $\sqrt[3]{(x^3 + 2x^2 - 7x - 64)}$, rational. Ans. $x = \frac{249408}{110935}$.

CASE II. When the first term of the formula is a cube, or when it is of the form

$$a^3x^3 + bx^2 + cx + d = \text{a cube}.$$

Assume $a^3x^3 + bx^2 + cx + d = (ax + r)^3 = a^3x^3 + 3a^2rx^2 + 3ar^2x + r^3$; where $3a^2r = b$, or $r = \frac{b}{3a^2}$; whence

$$cx + d = 3ar^2x + r^3,$$

$$\text{and} \quad x = \frac{r^3 - d}{c - 3ar^2}.$$

Or, substituting for r its value $\frac{b}{3ar}$, as above,

$$\text{we have } x = \frac{b^3 - 27da^6}{9a^3(3ca^3 - b^2)}.$$

Find such a value of x, as will make $x^3 - 3x^2 + 2x - 4$, a cube.

Here $r = -1$; $\therefore\ x^3 - 3x^2 + 2x - 4 = (x - 1)^3 =$

$$x^3 - 3x^2 + 3x - 1;$$

$$\text{whence } 2x - 4 = 3x - 1,$$

$$\text{or} \quad x = -3.$$

1. Find such a value of x, as will make $8x^3 - 4x^2 - 5x - 6$, a cube. Ans. $x = -\frac{161}{153}$.

2. Find such a value of x, as will make $27x^3 - 36x^2 + x - 18$, a cube. Ans. $x = -\frac{28}{27}$.

3. Find such a value of x, as will make $x^3 - 7x^2 + 3x + 2$, a cube. Ans. $x = \frac{289}{360}$.

4. Find such a value of x, as will make $8x^3 - 16x^2 + 12x - 17$, a cube. Ans. $x = \frac{395}{36}$.

CASE III. When both the first and last terms of the formula are cubes, or when it is of the form,

$$a^3x^3 + bx^2 + cx + d^3 = \text{a cube};$$

$$\text{assume } a^3x^3 + bx^2 + cx + d^3 = (ax + d)^3 =$$

$$a^3x^3 + 3a^2dx^2 + 3ad^2x + d^3; \text{ whence}$$

$$bx^2 + cx = 3a^2dx^2 + 3ad^2x,$$

$$\text{or } bx + c = 3a^2dx + 3ad^2;$$

$$\therefore \qquad x = \frac{3ad^2 - c}{b - 3a^2d}.$$

Find such a value of x, as will make $8x^3 + 24x^2 - 36x + 1$, a cube.

Here $8x^3 + 24x^2 - 36x + 1 = (2x + 1)^3 = 8x^3 + 12x^2 + 6x + 1$; whence

$$12x^2 + 6x = 24x^2 - 36x,$$

$$\text{or} \qquad x = 3\tfrac{1}{2}.$$

1. Find such a value of x, as will make $x^3 - 2x^2 + 3x + 8$, a cube. Ans. $x = -\frac{3}{4}$.

2. Find such a value of x, as will make $27x^3 - 36x^2 + x + 1$, a cube. Ans. $x = -\frac{8}{63}$.

3. Find such a value of x, as will make $8x^3 + x^2 - 15x - 1$, a cube. Ans. $x = \frac{21}{13}$.

4. Find such a value of x, as will make $64x^3 - 240x^2 + 29x - 27$, a cube. Ans. $x = -\frac{79}{96}$.

NOTE. This case obviously includes the two preceding cases, and if we make use of the formula of these two cases also, we will obtain other two values of x which will answer the conditions of the question.

When neither the first nor the last term of the formula is a cube, we have no direct method of solution; but if we can find a value of x by trial which satisfies the conditions of the question, another value may be found; thus,

If p be a value of x found by trial, then

$$ap^3 + bp^2 + cp + d = q^3$$

Assume $x = y + p$, and we obtain

$$\begin{array}{lllll}
ax^3 = & ay^3 + & 3apy^2 + & 3ap^2y + & ap^3 \\
bx^2 = & & by^2 + & 2bpy + & bp^2 \\
cx = & & & cy + & cp \\
d = & & & + & d \\
\hline
\end{array}$$

$$ay^3 + \left.\begin{array}{r} 3ap \\ + \; b \end{array}\right| y^2 + \left.\begin{array}{r} 3ap^2 \\ + \; 2bp \\ + \; c \end{array}\right| y + q^3 = \text{a cube}$$

Here the last term is a cube, and by Case I. we obtain a value of y, and hence that of x.

Find such a value of x, as will make $2x^3 - 3x + 7$, a cube.

Here we find on trial, that -1 is such a value of x, as will make the expression a cube; then let $x = y - 1$, and $2x^3 - 3x + 7 = 2(y-1)^3 - 3(y-1) + 7 = 2y^3 - 6y^2 + 3y + 8$, the last term of which is a cube; therefore, by Case I., $2y^3 - 6y^2 + 3y + 8 = (\frac{1}{4}y + 2)^3 = \frac{1}{64}y^3 + \frac{3}{8}y^2 + 3y + 8$;

whence $2y^3 - 6y^2 = \frac{1}{64}y^3 + \frac{3}{8}y^2$;

or $128y - 384 = y + 24$.

$$\therefore \; y = \frac{408}{127}, \text{ and } x = \frac{408}{127} - 1 = \frac{281}{127}.$$

And if we take this last value of x and proceed in the same way, we will obtain another value, and so on.

1. Find such a value of x, as will make $\sqrt[3]{(3x^3 + 3x^2 + 12x + 4)}$, rational.

Ans. $x = 2$ answers the conditions of the question; whence, by Case I., x also $= -\frac{144}{67}$.

2. Find such a value of x, as will make $\sqrt[3]{(2x^3 - 5x^2 + x + 3)}$, rational.

Ans. $x = 1$ satisfies the expression; whence, by Case I., x also $= -\frac{1}{3}$.

3. Find such a value of x, as will make $3x^3 - 7x^2 + 2x + 3$, a cube.

Ans. $x = 3$ satisfies the question; hence also $x = \frac{14793}{9872}$.

4. Find such a value of x, as will make $5x^3 - 10x^2 + 3x + 2$, a cube.

Ans. $x = 2$ satisfies the conditions; hence also $x = \frac{3526}{3527}$.

OF DOUBLE AND TRIPLE EQUALITIES.

When a single formula, containing one or more than one unknown quantity, is to be transformed into a complete power, such as a square or a cube, as in the preceding problems, it is called a *simple equality;* but when two or more formulæ, containing the same unknown quantity or quantities, are to be each transformed into some complete power, then they are called *double equalities, triple equalities,* and so on. The following are the only cases which admit of direct solution.

Case I. When the unknown quantity does not exceed the first degree, as

$$ax + b = \square$$
$$cx + d = \square.$$

Assume the first $ax + b = m^2$, and the second $cx + d = n^2$.

Then by equating the two values of x as found from these equations we obtain

$$\frac{m^2 - b}{a} = \frac{n^2 - d}{c}, \text{ or } cm^2 - cb = an^2 - ad.$$

Multiplying both sides of this by c, and transposing, we have

$$c^2m^2 = can^2 - cad + c^2b;$$

where n must be of such a value as to render the expression a square, and this value, when the question is resolvable, is found by Problem I., page 173.

Find a number such, that if either 128 or 192 be added to it, the sums shall be both squares.

Here let $x + 128 = m^2$, and $x + 192 = n^2$, then equating the two values of x as found from these equations, we have

$$m^2 - 128 = n^2 - 192;$$
$$\text{or } m^2 + 64 = n^2;$$

and since the quantity on the right hand side of the equation is a square, we have only to make $m^2 + 64$ a square.

Assume its root $= m + r$; then $m^2 + 64 = m^2 + 2mr + r^2$.

$$\therefore\ 2mr + r^2 = 64;$$
$$\text{and } m = \frac{64 - r^2}{2r}.$$

Where r may be any number at pleasure, take $r = 2$; then $m = \frac{64 - 4}{4} = \frac{60}{4} = 15$.

$$\therefore\ x = m^2 - 128 = 225 - 128 = 97.$$

Find a number such that if unity be either added to it, or subtracted from it, the sum and remainder shall be both squares.

$$\text{Here let } x + 1 = m^2, \text{ or } x = m^2 - 1$$
$$x - 1 = n^2, \text{ or } x = n^2 + 1;$$
$$\therefore\ m^2 - 1 = n^2 + 1$$
$$\text{or } m^2 - 2 = n^2.$$

In order to make $m^2 - 2$ a square, assume its root $= m - r$, then $m^2 - 2 = m^2 - 2mr + r^2$;

$$\text{or } -2mr + r^2 = -2;$$
$$\therefore\ m = \frac{r^2 + 2}{2r}.$$

Where r may be any number at pleasure, taking $r = 2$, we have $m = \frac{4 + 2}{4} = \frac{6}{4} = \frac{3}{2}$, consequently $x = \frac{9}{4} - 1 = \frac{5}{4}$.

Case II. When the unknown quantity does not exceed the second degree, and is found in each of the terms of the two formula; as,

$$ax^2 + bx = \square$$
$$cx^2 + dx = \square.$$

Assume $x = \frac{1}{y}$; then by substituting and multiplying each of the resulting expressions by y^2, we obtain

$$a + by = \square,$$
$$c + dy = \square,$$

from which the value of y, and thence of x, may be found by Case I.

Find a number such, that if it be either added to or subtracted from its square, the sum and remainder shall be both squares.

$$\text{Here } x^2 + x = \square,$$
$$\text{and } x^2 - x = \square.$$

Assume $x = \frac{1}{y}$; then $\frac{1}{y^2} + \frac{1}{y} = \square$, and $\frac{1}{y^2} - \frac{1}{y} = \square$; reducing, we have $1 + y = \square$, and $1 - y = \square$; make $1 + y = m^2$, or $y = m^2 - 1$, then $1 - y = 2 - m^2$, which is also to be made a square; but as neither the first nor the last term of this formula is a square, we must find a simple value of m by trial, which will make it rational; this is obviously 1.

Let $m = 1 - r$; then, by Case VII. Prob. I. we have $1 - y = 2 - m^2 = 2 - (1 - r)^2 = 1 + 2r - r^2$, and $y = r^2 - 2r$; or assuming $1 + 2r - r^2 = (1 - sr)^2 = 1 - 2sr + s^2r^2$;

$$\text{whence } 2 - r = 2s + s^2r;$$
$$\text{or} \quad r = \frac{2s - 2}{s^2 + 1},$$

and $x = \frac{1}{y} = \frac{1}{r^2 + 2r} = \frac{(s^2 + 1)^2}{4s - 4s^2}$, where s may be taken = any proper fraction; if $s = \frac{1}{2}$, then $x = \frac{25}{24}$.

When the double equality is of the form

$$ax^2 + bx + c = \square$$
$$dx^2 + ex + f = \square,$$

it is necessary first to resolve one of these equalities by Problem I., and then to substitute the value of x so found in the other equality, which will in consequence rise to the fourth degree, and, when possible, the solution will be found by Problem III. page 184.

Case III. When the equality is triple; as,

$$ax + by = \square$$
$$cx + dy = \square$$
$$ex + fy = \square.$$

Let $ax + by = u^2$, $cx + dy = v^2$, and $ex + fy = z^2$. Then, by first eliminating x in each of these equations, and then y in the two resulting equations, we shall have

$$(af - be)\,v^2 - (cf - de)\,u^2 = (ad - bc)\,r^2;$$

or assuming $v = ur$, and reducing the terms, the result gives the simple equality

$$\frac{af - be}{ad - bc} r^2 - \frac{cf - de}{ad - bc} = \frac{r^2}{u^2},$$

where the right hand term is a square, and it is therefore only necessary to find such a value of r as will make the left hand member a square, and this, when possible, is effected by Problem I.

After finding the value of r, we then have $v = ur$, and the first two equations give

$$x = \frac{d - br^2}{ad - bc} u^2, \text{ and } y = \frac{ar^2 - c}{ad - bc} u^2,$$

where u may be any number whatever.

When the three formula of the triple equality contain only one variable quantity, the simple equality to which it is necessary to reduce them rises to the fourth degree, and its resolution is effected by Problem III.

These are the only direct methods of solution, and therefore, in other cases of this kind, all that can be done is to determine successively by these methods several answers, when one is already found by trial; and if none of these succeed in solving the problem, it can only be done by some artifice of substitution which will fulfil one or more of its conditions, and then the remaining formula, when they are possible, may be resolved by some of the preceding methods.

Find three numbers in arithmetical progression such, that the sum of any two of them shall be a square.

Let the numbers sought be x, $x+y$, and $x+2y$, and make

$$2x + y = u^2$$
$$2x + 2y = v^2$$
$$2x + 3y = z^2;$$

then, by eliminating x and y from each of these equations, we have

$$v^2 - u^2 = z^2 - v^2;$$
$$\text{or } 2v^2 - u^2 = z^2;$$

putting $v = ur$, we have $2u^2r^2 - u^2 = z^2$, and dividing by u^2, $2r^2 - 1 = \frac{z^2}{u^2}$, where the right hand term is a square, it therefore only remains to make $2r^2 - 1$ a square, which it obviously is when $r = 1$.

But as this value upon trial does not answer the conditions of the problem, let $r = 1 - p$, then $2r^2 - 1 = 2(1-p)^2 - 1 = 1 - 4p + 2p^2 = \square$.

Assume $1 - 4p + 2p^2 = (1-np)^2 = 1 - 2np + n^2p^2$;

$$\text{or } -4 + 2p = -2n + n^2p;$$

whence $p = \frac{2n-4}{n^2-2}$, and $r = 1 - \frac{2n-4}{n^2-2} = \frac{n^2 - 2n + 4}{n^2-2}$, where n may be any number whatever. After finding the value of r, the value of x and y are found from the equations

$$y = v^2 - u^2 = u^2r^2 - u^2 = (r^2 - 1)\, u^2$$
$$x = \tfrac{1}{2}(u^2 - y = \tfrac{1}{2}(2 - r^2)\, u^2.$$

Now, in order that x and y may be positive, it is evident that r must be taken greater than 1 and less than $\sqrt{2}$.

If therefore $n = \frac{9}{5}$ we have $r = \frac{41}{31}$;

whence $x = \left(2 - \frac{1681}{961}\right) \times \frac{u^2}{2}$, and $y = \left(\frac{1681}{961} - 1\right) u^2$; or taking $u = 2 \times 31$, $x = 482$, and $y = 2880$; $\therefore\ x = 482$, $x + y = 3362$, and $x + 2y = 6242$, the numbers required.

To divide a given square into two squares.

Let a^2 be the given square, and let $x^2 =$ one of the squares, then $a^2 - x^2 =$ the other.

Now, in order to make $a^2 - x^2 = \square$, suppose its root $= rx - a$, then $a^2 - x^2 = r^2x^2 - 2arx + a^2$, or

$$-x^2 = r^2x^2 - 2arx,$$

whence $x = \frac{2ar}{r^2+1} =$ the root of the first part,

and $rx - a = \frac{2ar^2}{r^2+1} - a = \frac{ar^2 - a}{r^2+1} =$ the root of the second.

Hence $\left(\frac{2ar}{r^2+1}\right)^2$, and $\left(\frac{ar^2-a}{r^2+1}\right)^2$ are the squares required, where a may be any square, and n any number whatever greater than 1.

To divide a number consisting of two squares into two *other* squares.

Let a^2 and b^2 be the squares of which the given number consists, let $x + a =$ side of one of the squares sought, and the other any number of times x, lessened by b, as $rx - b$; then $x^2 + 2ax + a^2 + r^2x^2 - 2rxb + b^2 = a^2 + b^2$, or

$$r^2x^2 + x^2 = 2axb - 2ax;$$
$$\text{or } r^2x + x = 2rb - 2a;$$
$$\therefore \quad x = \frac{2rb - 2a}{r^2 + 1},$$

and $x + a = \frac{2rb - 2a}{r^2 + 1} + a = \frac{2rb - a(r^2 - 1)}{r^2 + 1}$, the side of the one square, and $rx - b = \frac{r(2rb - 2a)}{r^2 + 1} - b = \frac{2b(r^2 - 1) - 2ar}{r^2 + 1}$, the side of the other.

To find two square numbers that shall have a given difference.

Let $d =$ the given difference, and x^2 the least square, then $x^2 + d$ the greater square, which must be a square; let it be $= (x+a)^2 = x^2 + 2ax + a^2$, then $2xa = d - a^2$, and $x = \frac{d}{2a} - \frac{a}{2}$, where a may be taken at pleasure; therefore, let $ab = d$, then $x = \frac{b - a}{2}$, and $x + a = \frac{b + a}{2}$, the sides of the two squares.

Suppose $d = 24$, and $ab = 12 \times 2$, then $x = \frac{12 - 2}{2} = 5$, and $x + 2 = \frac{12 + 2}{2} = 7$; now, $7^2 - 5^2 = 49 - 25 = 24$, the given difference.

To find two numbers such, that, if either of them be added to the square of the other, the sum shall be a square.

Let x be one of the numbers, and when its square is added to the other number, let the square produced be $(x - a)^2 = x^2 - 2ax + a^2 = x^2 + y$, whence $x = \frac{a^2 - y}{2a}$, and $y^2 + \frac{a^2 - y}{2a}$ is a square, suppose it $= (y + b)^2 = y^2 + 2by + b^2$, whence $y = \frac{a^2 - 2ab^2}{4ab + 1}$, and consequently $x = \frac{2a^2b + b^2}{4ab + 1}$, where

a and b may be taken at pleasure, but so that a may be greater than $2b^2$.

To find two numbers such, that their sum and difference shall be squares.

Let x and y be the numbers, then $x+y$ and $x-y$ are squares, and their difference is $2y$; find therefore two squares of which the difference is $2y$, namely, $(\frac{y}{2}+1)^2$ and $(\frac{y}{2}-1)^2$, then $(\frac{y}{2}+1)^2 = \frac{y^2}{4}+y+1 = x+y$, and $\frac{y^2}{4}+1 = x$, where y may be taken at pleasure.

To find three numbers such, that the sum of any two of them, and also the sum of the three, shall be squares.

Let $2ax$ be the first, $x^2 - 2ax$ the second, and $ax+\frac{a^2}{4}$ the third; then the sum of the first and second is x^2, that of the second and third $x^2 - ax + \frac{a^2}{4}$, and the sum of the three $x^2+ax+\frac{a^2}{4}$, all of them complete squares; it therefore remains to make the sum of the first and third $3ax+\frac{a^2}{4}$ a square; let it then be b^2, hence $x = \frac{4b^2 - a^2}{12a}$, where a and b may be taken at pleasure. Let $b=4$, and $a=2$, then $x = \frac{64-4}{24} = \frac{5}{2}$, $2ax = 10$, $x^2 - 2ax = -\frac{15}{4}$, $ax+\frac{a^2}{4} = 6$; hence the sum of the first and second is $\frac{25}{4}$, of the first and third $\frac{64}{4}$, of the second and third $\frac{9}{4}$, and of the whole three $\frac{49}{4}$, all squares.

To find three squares such, that the sum of any two of them shall be a square.

Let x^2, y^2, z, $=$ the numbers, and first let $z=1$, then $x^2+1 =$ a square $= (x-m)^2 = x^2 - 2xm + m^2$, or $x = \frac{m^2-1}{2m}$; also $y^2+1 =$ a square $= (y-n)^2 = y^2\ 2yn +$

n^2, or $y = \frac{n^2 - 1}{2n}$; therefore $\left(\frac{m^2 - 1}{2m}\right)^2 + \left(\frac{n^2 + 1}{2n}\right)^2 =$ a square. To render this as simple as possible, suppose $m - 1 = n + 1$, then $\left(\frac{m^2 - 1}{2m}\right)^2 + \left(\frac{(m-3)(m-1)}{2(m-2)}\right)^2$ is a square, or $\frac{m+1}{m^2} + \frac{(m-3)^2}{(m-2)^2} =$ a square, reducing this to a common denominator, then $(m+1)^2 \times (m-2)^2 + m^2 \times (m-3)^2$ is a square, or $2m^4 - 8m^3 + 6m^2 + 4m + 4 =$ a square $= (rm^2 + m + 2)^2 = r^2m^4 + 2rm^3 + (4r + 1)\, m^2 + 4m + 4$. Now, if $4r + 1$ be taken $= 6$, then $r = \frac{5}{4}$, and the equation becomes $2m^4 - 8m^3 + 6m^2 + 4m + 4 = \frac{25}{16} m^4 + \frac{10}{4} m^3 + 6m^2 + 4m + 4$,

$$\text{or } 2m^4 - 8m^3 = \frac{25}{16} m^4 + \frac{10}{4} m^3;$$

$$\therefore 2m - 8 = \frac{25}{16} m + \frac{10}{4};$$

$$\text{whence } m = 24,$$

and since $m - 1 = n + 1$, $n = 22$;

therefore $x = \frac{m^2 - 1}{2m} = \frac{575}{48}$, $y = \frac{n^2 - 1}{2n} = \frac{483}{44}$, and $z = 1$, and the numbers are $\frac{330625}{2304}$, $\frac{233289}{1936}$, and 1.

To divide a given cube into three cubes.

Let $a^3 =$ given cube. Take any cube less than $\frac{a^3}{2}$, such as b^3, and let the other two cubes be $(a - x)^3$ and $\left(\frac{a^2 x}{b^2} - b\right)^3$. Hence the sum of these two, or $a^3 - b^3 = a^3 + \left(3a - \frac{3a^4}{b^3}\right) x^2 + \left(\frac{a^6}{b^6} - 1\right) x^3 - b^3$, or $\frac{a^6 - b^6}{b^6} x = \frac{3a^4 - 3ab^3}{b^3}$; whence $x = \frac{3a^4b^3 - 3ab^6}{a^6 - b^6} = \frac{3ab^3}{a^3 + b^3}$, conse-

quently the three roots are b, $a\frac{a^3-2b^3}{a^3+b^3}$, and $b\frac{2a^3-b^3}{a^3+b^3}$, where a and b may be any numbers whatever.

Let $a=3$ and $b=1$, then the first cube is 1^3, the second $\left(\frac{75}{28}\right)^3$, and the third $\left(\frac{53}{28}\right)^3$, and their sum is $\frac{21952}{21952}+\frac{421875}{21952}+\frac{148877}{21952}=\frac{592704}{21952}=\left(\frac{84}{28}\right)^3$.

Otherwise,

Let a^3 be the given cube, and let $a=v+z$. Let one of the required roots $=v-z$, and let the roots of the other two required cubes be $x+y$ and $x-y$. Then the difference of the two first cubes will be $2z\times(z^2+3v^2)$, and the sum of the other two will be $2x(x^2+3y^2)$. Here x^2+3y^2 may be resolved into two factors $(m^2+3n^2)\times(t^2+3u^2)$, for making $x=mt+3nu$, and $y=nt-mu$, we will have $x^2=m^2t^2+6mntu+9n^2u^2$, and $3y^2=3n^2t^2-6mntu+3m^2u^2$, therefore $x^2+3y^2=m^2t^2+3n^2t^2+3m^2u^2+9n^2u^2=(m^2+3n^2)\times(t^2+3u^2)$.

In like manner, z^2+3v^2 may be resolved into the two factors $(a^2+3b^2)\times(t^2+3u^2)$, and since $2z\,(z^2+3v^2)=2x\,(x^2+3y^2)$, therefore $2(mt+3nu)\times(m^2+3n^2)\times(t^2+3u^2)=2(at+3bu)\times(a^2+3b^2)\times(t^2+3u^2)$; whence $(mt+3nu)\times(m^2+3n^2)=(at+3bu)\times(a^2+3b^2)$, or $mt\times(m^2+3n^2)-at\times(a\ +3b^2)=3bu\times(a^2+3b^2)-3nu\times(m^2+3n^2)$; whence $t=3b\times(a^2+3b^2)-3n\times(m^2+3n^2)$, and $u=m\times(m^2+3n^2)-a\times(a^2+3b^2)$. Having thus found the value of t and u in terms of any values of m, n, a, and b, we can find $x=mt+3nu$, $y=nt-mu$, $v=at+3bu$, and $z=bt-au$, after which the numbers will be $x+y$, $x-y$, and $v-z$, and the root of the given cube will be $v+z$ for a^3, the cubes will be $\frac{(x+y)^3a^3}{v+z^3}$, $\frac{(x-y)^3a^3}{(v+z)^3}$, and $\frac{(v-z)^3a^3}{(v+z)^3}$.

Let $m=2$, $n=1$, $a=1$, and $b=2$, then $m^2+3n^2=4+3=7$, and $a^2+3b^2=1+3\times4=13$; therefore

$t = 3 \times 2 \times 13 - 3 \times 1 \times 7 = 78 - 21 = 57$, and $u = 2 \times 7 - 1 \times 13 = 14 - 13 = 1$, whence $x = 2 \times 57 + 3 \times 1 \times 1 = 114 + 3 = 117$, $y = 1 \times 57 - 2 \times 1 = 55$, $v = 1 \times 57 + 3 \times 2 \times 1 = 63$, and $z = 2 \times 57 - 1 \times 1 = 113$; wherefore $x + y = 172$, $x - y = 62$, $z - v = 50$, and $z + v = 176$, or taking their halves, the numbers are 86, 31, 25, and 88, so that $86^3 + 31^3 + 25^3 = 88^3$. If the given cube be not 88^3 but 4^3, then the numbers are $\frac{86^3}{22^3} + \frac{31^3}{22^3} + \frac{25^3}{22^3} = \frac{88^3}{22^3} = 4^3$.

To find four numbers such, that if a given square be added to the product of any two of them, the sum shall be a square.

Let a^2 be the given square, and $a^2 + 2ax + x^2$, the square arising by adding the product of the first and second to a^2; then if x be the first, $2a + x =$ the second. Again, let $a^2 + 4ax + 4x^2$ be the square arising by adding the product of the first and third to a^2, then the third is $4a + 4x$. Next, let $a^2 + 6ax + 9x$ be the square arising by adding the product of the first and fourth to a^2, then the fourth is $6a + 9x$. Likewise $(a^2 + 2a + x)(4a + 4x) = (3a + 2x)^2$, and $a^2 + (4a + 4x)(6a + 6x) = (5a + 6x)^2$. Also $a^2 + (2a + x) \times (6a + 9x)$, or $13a^2 + 24ax + 9x^2$ should be a square, suppose it $= (ma - 3x)^2 = m^2a^2 - 6max + 9x^2$; whence $13a^2 + 24ax = m^2a^2 - 6max$, or $x = \frac{m^2a - 13a}{6m + 24}$, whence the rest of the numbers are $\frac{m^2a + 12ma + 35a}{6m + 24}$, $\frac{4m^2a + 24ma + 44a}{6m + 24}$, and $\frac{9m^2a + 36ma + 27a}{6m + 24}$, where a and m may be any numbers whatever.

If we take $a = 10$, and $m = 11$, the first is $\frac{121 \times 10 - 130}{66 + 24} = 12$, the second is $\frac{121 \times 10 + 12 \times 11 \times 10 + 35 \times 10}{66 + 24} = 32$, the third is $\frac{4 \times 121 \times 10 + 24 \times 11 \times 10 + 44 \times 10}{66 + 24} = 88$, and the fourth is $\frac{9 \cdot 121 \cdot 10 + 36 \cdot 11 \cdot 10 + 27 \cdot 10}{66 + 24} = 168$.

1. To find two numbers such, that their sum and also the square of either added to the other shall be squares.

Ans. Any two fractions of which the sum is $\frac{1}{4}$.

2. To find three square numbers in arithmetical progression. Ans. 1, 25, and 49.

3. To find three numbers such, that the square of any one of them added to the other shall be a square.

Ans. Any three fractions of which the sum is $\frac{1}{4}$.

4. To find two numbers such, that their difference shall be equal to the difference of their squares, and the sum of their squares shall be a square. Ans. $\frac{3}{7}$ and $\frac{4}{7}$.

5. To find three numbers in geometrical progression such, that each of them increased by a given number (21) shall be a square. Ans. 100, 4, $\frac{4}{25}$.

6. To find two numbers such, that their product added to the sum of their squares shall be a square. Ans. 3 and 5.

7. To divide the number 10 into four parts, so that the sum of any three of them shall be a square.

Ans. 1, 6, $\frac{186}{289}$, and $\frac{681}{289}$.

8. To find two numbers such, that their sum either increased or diminished by their difference, or by the difference of their squares, shall be a square. Ans. $\frac{49}{50}$ $\frac{1}{50}$

9. To find two numbers such, that if unity be added to each of them, and also to their sum or difference, the sums shall be squares. Ans. 120 and 168.

10. To find three numbers such, that if each of them be added to the sum of their squares the sums will be squares.

Ans. $\frac{2209}{31460}$, $\frac{6627}{31460}$, and $\frac{19881}{62920}$.

11. To find three numbers such, that if their sum be either added to or subtracted from the square of each, the sums and remainders shall be squares. Ans. $\frac{518}{96}$, $\frac{406}{96}$, and $\frac{791}{96}$.

12. To find three squares such, that the sum of their squares shall be a square. Ans. 4, $\frac{9}{4}$, and $\frac{36}{25}$.

13. To find three square numbers such, that the difference of any two of them shall be a square.

Ans. 153^2, 185^2, and 697^2.

14. Given two cubes, a^3 and b^3, to find two other cubes, of which the difference shall be equal to the sum of the given cubes, viz. a^3+b^3. Ans. $\left(a\frac{a^3+2b^3}{a^3-b^3}\right)^3$ and $\left(b\frac{2a^3+b^3}{a^3-b^3}\right)^3$

15. Given two cubes, 1 and 8, to find a third cube so that the sum of the three shall be a cube. Ans. $\left(\frac{20}{7}\right)^3$.

16. To find three cubes such, that if from each of them a given number (1) be taken, the sum of the remainders shall be a square. Ans. 8, $\frac{854670349}{884736}$, and $\frac{8365427}{884736}$.

17. To find three numbers such, that if the cube of their sum be added to each, the three sums shall be cubes.

Ans. $\frac{63}{12167}$, $\frac{124}{12167}$, and $\frac{342}{12167}$.

18. To find three numbers in arithmetical progression such, that the sum of their cubes shall be a cube.

Ans. 149, 256, and 363.

19. To find three cube numbers such, that their sum shall be a cube. Ans. 9^3, 12^3, 15^3.

20. To find two numbers such, that their sum shall be equal to the sum of their cubes. Ans. $\frac{5}{7}$ and $\frac{8}{7}$.

21. To find four numbers such, that the square of the greatest shall be equal to the squares of the other three, and that they shall be the least integers. Ans. 7, 6, 3, 2.

22. To find two numbers such, that the sum of their squares, and also the sum of their cubes, shall be squares.

Ans. 273 and 364.

23. To find two numbers such, that their product and quotient shall each be a square, and their sum a cube.

Ans. 25 and 100.

24. To find three numbers such, that both the sum and the difference of every two of them shall be squares.

Ans. $\frac{153^2+185^2-697^2}{2}$, $\frac{697^2+153^2-185^2}{2}$, and $\frac{697^2+185^2-153^2}{2}$.

25. To find three numbers such, that the square of each of them added to the product of the other two shall be a square.

Ans. 2, $\frac{257}{16}$, and $\frac{129}{64}$.

26. To find two numbers such, that if each of them be added to their product the sums shall be squares.
Ans. 9 and 360.

27. To find three squares in arithmetical progression.
Ans. 49, 289, and 529.

28. To find three numbers such, that the square of each added to the sum of the other two shall be a square.

Ans. $\frac{5}{9}$, $\frac{12}{9}$, and $\frac{4}{9}$.

29. To find two numbers such, that each of their squares added to their product shall be a square. Ans. 9 and 16.

30. To find two numbers such, that if 1 be taken from the sum of their squares, and also from the difference of their squares, the remainders shall be squares. Ans. 8 and 9.

31. To find three numbers such, that their sum shall be a square, and if the sum of their squares be added to each of the numbers, the sums shall be squares.

Ans. $\frac{26}{2401}$, $\frac{78}{2401}$, and $\frac{39}{2401}$.

32. To find three numbers in geometrical progression such, that any two of them in order taken together shall be a square. Ans. 40, 360, and 3240.

33. To find three numbers such, that if their sum be either added to or subtracted from the square of each, the sums and remainders shall be squares. Ans. $\frac{513}{96}$, $\frac{406}{96}$, and $\frac{791}{96}$.

34. To find three square numbers such, that if from the sum of any two of them the third be subtracted, the remainder shall be a square. Ans. 241^2, 149^2, and 269^2.

35. To find two numbers such, that their sum shall be a square, and that the square of each added to the other shall also be a square. Ans. $\frac{1}{6}$ and $\frac{1}{12}$.

36. To find two square numbers such, that their sum shall be a square, and their difference shall be the cube root of that square. Ans. $\frac{784}{15625}$ and $\frac{441}{15625}$.

37. To find three numbers such, that if each be subtracted from the cube of their sum, the remainders shall be cubes.

Ans. $\frac{13851}{85184}$, $\frac{18954}{85184}$, and $\frac{19467}{85184}$.

38. To find three numbers such, that if the cube of their sum be subtracted from each, the remainders shall be cubes.

Ans. $\frac{82349}{729000}$, $\frac{59914}{729000}$, and $\frac{26837}{729000}$.

39. To find three numbers such, that the excess of the greatest above the middle one shall be 3 times the excess of the middle one above the least, and also that the sum of any two of them shall be a square. Ans. 58, 1878, and 7338.

40. To divide the fraction $\frac{1}{4}$ into 3 parts such, that each of them diminished by the cube of their sum shall leave a square. Ans. $\frac{250}{1600}$, $\frac{61}{1600}$, and $\frac{89}{1600}$.

OF PERMUTATIONS AND COMBINATIONS.

PERMUTATIONS of things are the different orders in which they can be arranged; thus the permutations of the three letters *a, b, c,* taken two and two together, are *ab, ba; ac, ca; bc, cb.*

COMBINATIONS of things are the different varieties which can take place in arranging them, by taking them two and two together, three and three together, &c.; thus the combinations that can be made of the three letters *a, b, c,* taken two and two together, are *ab, ac, bc.*

PROP. I. The number of permutations that can be formed out of n things, taken two and two together, is $n(n-1)$; taken three and three together is $n(n-1)(n-2)$; and taken s and s together, is $n(n-1)(n-2)\ldots(n-\overline{s-1})$.

Let $a, b, c, d,$ &c., be the n things; it is obvious that a may be placed before each of the rest, and they form $(n-1)$ permutations, $ab, ac, ad,$ &c. In like manner there are $(n-1)$ permutations in which b is placed first, and so on of the rest; and since there are n things, it is manifest that the whole number of permutations, when taken two and two together, is $n(n-1)$.

Again, if we take $(n-1)$ things, $b, c, d,$ &c., the whole number of permutations taken two and two together is $(n-1)(n-2)$. Now, if we place a before each of these, there are $(n-1)(n-2)$ permutations, when taken three and three together, in which a stands first; and since there are also the same number of permutations when each of the others stands

first, consequently the whole number of permutations, taken three and three together, is $n(n-1)(n-2)$.

In the same way it may be shown, that when the things are taken four and four together, there are $n(n-1), (n-2), (n-3)$ permutations, and so on; therefore, if taken s and s together, the whole number of permutations is $n(n-1), (n-2), \ldots (\overline{n-s-1})$.

Cor. If all the things are taken together, the number of permutations is $n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1$. Thus the number of permutations which can be formed of 7 things is $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$.

Prop. II. If the same thing occurs t times, then the number of permutations in n things taken altogether is $\dfrac{n(n-1)\ldots 2 \cdot 1}{1 \cdot 2 \ldots t}$.

It is evident, that if instead of the same thing occurring t times, there had been t different things, they would have formed $1 \cdot 2 \cdot 3 \ldots t$ permutations from their interchange with each other; but when these things are all alike there can be no interchange of place among them, and these permutations are all reduced to one. Now, since this is true, for every position of the other things there will therefore be $1 \cdot 2 \cdot 3 \ldots t$ times fewer permutations when t quantities are alike than when they are all different.

Prop. III. If the same thing occurs t times, another r times, and a third s times, and so on, the number of permutations will be $\dfrac{n(n-1)(n-2)\ldots\ldots 2 \cdot 1}{1 \cdot 2 \ldots t \times 1 \cdot 2 \ldots r \times 1 \cdot 2 \ldots s}$.

It was shown in last Prop. that if t things be alike, there will be $1 \cdot 2 \ldots t$ times fewer permutations than if they were all different; so if there be also r other things alike, but different from the first, there will be $1 \cdot 2 \ldots r$ times fewer permutations, and so on; consequently there will be altogether $1 \cdot 2 \ldots t \times 1 \cdot 2 \ldots r \times 1 \cdot 2 \ldots s$ times fewer permutations than when the things are all different.

Prop. IV. The number of combinations that can be formed out of n things taken two and two together, is $\dfrac{n(n-1)}{1 \cdot 2}$;

taken three and three together, is $\frac{n(n-1)(n-2)}{1.2.3}$; and when

taken s and s together, is $\frac{n(n-1)(n-2)\ldots(n-\overline{s-1})}{1.2.3\ldots s}$.

It has been shown (Prop. I.) that the number of permutations of n things taken two and two together is $n(n-1)$; but each combination of two things ab admits of 1×2 permutations ab, ba, hence there are twice as many permutations of n things taken two and two together as there are combinations.

In like manner, there are $n(n-1)(n-2)$ permutations in n things taken three and three together; and since each combination of three things abc admits of $1 . 2 . 3$ permutations, *abc, acb, cab, bac, bca, cba,* hence there are $1 . 2 . 3$ times as many permutations as combinations of n things, taking three and three together.

In like manner it is shown that there are $n(n-1)(n-2)\ldots(n-\overline{s-1})$ permutations of n things taken s and s together, and since each combination of s things admits of $1 . 2 . 3 \ldots s$ permutations, consequently the number of combinations of n things taken s and s together is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-\overline{s-1})}{1.2.3\ldots\ldots s}.$$

1. Required the number of permutations that can be made of the letters in the word *Edinburgh*? Ans. 362880.

2. Required all the permutations that can be made of the letters in the word *Waterloo*? Ans. 20160.

3. Required how many changes can be made with 14 men standing in a rank? Ans. 87178291200.

4. In how many different ways may a captain select three men out of his company of 100, so as to save five from the duty? Ans. 138415.

5. How many different hands can be held at the game of whist? Ans. 635013559600.

6. How many days can a company of 12 persons sit in a different position round a table at dinner?

Ans. 479001600 days.

7. A general requested, as a reward for his services, a farthing for every file of 10 men which he could take out of a company of 100 men. How much did this amount to?

Ans. £18031572350, 9s. 2d.

8. Required the number of combinations or conjunctions that can occur with the 7 satellites of Saturn, when taken two by two, three by three, &c., up to seven? Ans. 120.

9. How many changes can be made with 5 letters out of the 26 which compose the alphabet? Ans. 7893600.

10. How many changes can be rung on 24 bells?
Ans. 620448401733239439360000.

11. If the colonel of a regiment, with 30 officers, invite 6 each day to dinner, how long may it be before he invites the same company? Ans. 593775 days.

12. In how many different ways may a party of 10 men be drawn up? Ans. 3628800.

OF PROBABILITIES.

The probability that an event will occur may sometimes be inferred from the nature of the case; thus, if a die with six equal faces be thrown upon a table, we consider it equally probable that it shall present any of its faces, and therefore that there is one chance in six it shall present an ace.

In calculations of probabilities certainty is represented by unity, and any degree of probability by a fraction; thus, a die being thrown, the chance that it presents an ace is $\frac{1}{6}$, and the chances of not throwing an ace are $\frac{5}{6}$.

Prop. I. If an event may happen in a ways, and fail in b ways, any of these being equally probable, the chance of its happening is $\frac{a}{a+b}$, and the chance of its failing is $\frac{b}{a+b}$.

For the chance of its happening : chance of its failing :: $a : b$, therefore the chance of its happening : chance of its happening + chance of its failing :: $a : a+b$; but since it must either happen or fail, the chance of its happening + the chance of its failing is certainty, which is represented by unity.

Hence the chance of its happening is $\frac{a}{a+b}$. In like manner it may be shown, that the chance of its failing is $\frac{b}{a+b}$.

Prop. II. If two events be independent of each other, and a, a' be the number of ways in which they may happen, and

b, b' the number of ways in which they may fail, then the probability that they will both happen is $\frac{a}{a+b} \times \frac{a'}{a'+b'} = \frac{aa'}{(a+b)(a'+b')}$.

For every way a in which the first event can happen may be combined with every way a' in which the second event can happen, and thus form aa' combinations in which both can happen. Also, every way $a+b$ may be combined with every way $a'+b'$, and thus form altogether $(a+b)(a'+b')$ combinations, in which the events may either happen or fail. Hence the chance that both will happen is $\frac{aa'}{(a+b)(a'+b')}$. In like manner it may be shown that the chance that both will fail is $\frac{bb'}{(a+b)(a'+b')}$; that the first will happen and the second fail, the chance is $\frac{ab'}{(a+b)(a'+b')}$; and that the second will happen and the first fail the chance is $\frac{a'b}{(a+b)(a'+b')}$.

Prop. III. If there are three independent events, and the probabilities of their happening be respectively $\frac{a}{a+b}$, $\frac{a'}{a'+b'}$, and $\frac{a''}{a''+b''}$, then the probability that they will all happen is $\frac{aa'a''}{(a+b)(a'+b')(a''+b'')}$.

For since each of the ways a'', in which the third event may happen, may be combined with each of the aa' ways in which the two first can happen, therefore $aa'a''$ is the number of ways in which all the three can happen. Also every $a''+b''$ may be combined with $(a+b)(a'+b')$, and thus form altogether $(a+b)(a'+b')(a''+b'')$ combinations, or the number of ways in which the three events may either happen or fail, consequently the probability that all the three will happen is $\frac{aa'a''}{(a+b)(a'+b')(a''+b'')}$.

In the same way we may obtain formula for any number of events.

PROP. IV. If the probability that an event will happen in one trial be $\frac{a}{a+b} = r$, and the probability of its failing be $\frac{b}{a+b} = s$, then the probability of its happening exactly m times in n trials is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-m+1)}{1\,.\,2\,.\,3\,\ldots\ldots\,(n-3)} r^m s^{n-m}.$$

For the chance of its happening in any one particular trial being r, and the chance of its failing in all the other $n - 1$ trials being s^{n-1}, therefore the probability of its happening in one particular trial and failing in all the rest is rs^{n-1}; and since there are n trials, the chance that it will happen in some one of these and fail in all the rest is nrs^{n-1}. The chance that it will happen in two particular trials and fail in all the others is r^2s^{n-2}; and since there are $\frac{n(n-1)}{2}$ different ways in which it may happen twice in n trials and fail in all the others, consequently the probability that it will happen twice in n trials is $\frac{n(n-1)}{2} r^2s^{n-3}$. In like manner it may be shown that the probability of its happening three times is $\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3} r^3s^{n-3}$; of its happening four times is $\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4} r^4s^{n-4}$, and so on; wherefore the chance of its happening m times in n trials is $\frac{n(n-1)(n-2)\ldots\ldots(n-m+1)}{1\,.\,2\,.\,3\,\ldots\ldots\,m} r^m s^{n-m}$.

1. What is the chance of throwing either ace or six with a single die? Ans. $\frac{1}{3}$.

2. If 7 white balls, 6 black and 4 red, be put into a box, what is the chance of drawing out a red one? Ans. $\frac{4}{17}$.

3. If 5 white balls and 7 black be put into one box, and 1 white ball and 11 black ones into another, what is the chance of drawing a white ball from both? Ans. $\frac{5}{144}$.

4. A and B having gained one game at a rubber of whist, what is the chance that they gain the rubber?

Ans. $\frac{3}{4}$, the odds being 3 to 1 in their favour.

5. A bag contains 12 counters, of which 4 are white; what is the probability that precisely 3 white ones shall be drawn in 7 trials? Ans. [illegible]

6. A bag contains 2 white and 3 black balls; what are the chances which A and B have of first drawing out a white ball if A has the first trial? Ans. A $\frac{3}{5}$, and B $\frac{2}{5}$.

7. What is the chance of throwing at one trial, with two dice, four and ace? Ans. $\frac{1}{18}$.

8. A pack of 52 cards is divided according to suits into 4 equal heaps; what is the chance of drawing one of the three court cards of spades? Ans. $\frac{3}{52}$.

9. In how many trials may a person undertake for an even wager to throw aces with 2 dice? Ans. 24·6.

10. If 4 cards be drawn out of a pack of 52, what is the probability that each is of a different suit? Ans. $\frac{2197}{20825}$.

OF LIFE ANNUITIES.

The value of annuities on lives depends upon two circumstances,—the probability of the duration of life, and the rate of interest.

Prob. I. To find the probability that a person A will live any number of years.

Let the number of persons in the table of mortality of the given age be a, and the number of those at the end of 1, 2, 3, . . . n years be respectively $a_1, a_2, a_3, \ldots a_n$, then the number of equal chances the person has of living to the end of one year is a_1, since that is the number alive at the end of one year, and the whole number of chances of living, and of not living to the end of the year, is a, consequently the probability that he will live one year, is $\frac{a_1}{a}$, that he will live two years, is $\frac{a_2}{a}$, and that he will live n years, is $\frac{a_n}{a}$.

Let $\frac{a_1}{a}, \frac{a_2}{a}, \ldots \frac{a_n}{a}$, be represented by r_1, r_2, and r_n, then the probability that the person will be dead at the end of 1, 2, 3, . . . n years is $1 - r_1$, $1 - r_2$, $1 - r_3 \ldots 1 - r_n$.

1. What is the probability that a person aged 45 shall live 5 years? Ans. $\frac{2857}{3248}$.*

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* The Table used here is the Northampton one.

2. What is the probability that a person aged 40 shall live 15 years? Ans. $\frac{2448}{3635}$.

3. What is the probability that a person aged 30 shall live other 30 years? Ans. $\frac{2038}{4385}$.

PROB. II. To find the probability that two persons A and B will live any number of years.

Let, as in the last, $b_1, b_2, b_3, \ldots b_n$, be the number of equal chances which B has of living 1, 2, 3, ... n years, and let $s_1, s_2, s_3, \ldots s_n$, be the probability that he will live 1, 2, 3, ... n years; then since the probability that A will live n years is r_n, and that b will live n years is s_n; hence the probability that both will live n years is $r_n s_n$; that A will be alive and B dead is $r_n(1 - s_n)$; that B will be alive and A dead is $s_n(1 - r_n)$; that both will be dead is $(1 - r_n)(1 - s_n)$; and that one of them will be alive is $r_n + s_n - r_n s_n$.

What are all the probabilities of living and dying, at the end of 15 years, of two persons, the one now 20 and the other 25 years of age?

Ans. The probability that both will be alive is $\frac{1457635}{2442832}$; that only the younger will be alive is $\frac{451125}{2442832}$; that only the elder will be alive is $\frac{407847}{2442832}$; that both will be dead is $\frac{126225}{2442832}$; that one of them will be alive is $\frac{2316607}{2442832}$.

What are all the probabilities of living and dying, at the end of 20 years, of two persons aged 30 and 40 years?

Ans. The probability that both will be alive is $\frac{5822566}{15939475}$; that only the younger will be alive is $\frac{4562629}{15939475}$; that only the elder will be alive is $\frac{5114064}{15939475}$; that both will be dead is $\frac{2440216}{15939475}$; and that one of them will be alive is $\frac{15499259}{15939475}$.

In like manner it may be shown, that if t_n represent the probability that C will live n years, the probability that the three persons A, B, and C, shall all live n years is $r_n s_n t_n$; that A shall die and B and C live is $(1 - r_n) s_n t_n$; that B shall die and A and C live is $(1 - s_n) r_n t_n$; that C shall die and A and B live is $(1 - t_n) r_n s_n$; that at least two of the three shall live is the sum of these four probabilities, or $r_n s_n + r_n t_n + s_n t_n - 2 r_n s_n t_n$; that at least one of the three shall live is $1 - \{(1 - r_n)(1 - s_n)(1 - t_n)\}$; and the proba-

bility that the three shall die in n years is $(1 - r_n)(1 - s_n)(1 - t_n)$.

Prob. III. To find the present value of an annuity of £1 per annum during the life of A, at any rate of interest.

Let q be the rate per cent., then $\frac{1}{1+q}$ is the present value of £1 certain at the end of one year; let this be represented by p. Then, since r_1 is the probability that A shall live one year, the present value of £1 to be paid at end of the year, on the contingency of A's being alive, is $r_1 p$, the present values of the second, third, &c. years' annuities are respectively $r_2 p^2$, $r_3 p^3$, &c. Hence the present value of £1 during the whole life of the person is $r_1 p + r_2 p^2 + r_3 p^3 + r_4 p^4 +$ &c. to the end of the table.

To find the present value of an annuity of £1 on a life of 90, at 5 per cent.

Here from the table we have the probabilities $\frac{34}{46}$, $\frac{24}{46}$, $\frac{16}{46}$, $\frac{9}{46}$, $\frac{4}{46}$, and $\frac{1}{46}$, and p and its powers are ·9524, ·9070, ·8638, ·8227, ·7835, and ·7462, whence the present value of the annuity is $\frac{1}{46}$ (·9524 × 34 + ·9070 × 24 + ·8638 × 16 + ·8227 × 9 + ·7835 × 4 + ·7462) = $\frac{79{\cdot}2549}{46}$ = £1·7229.

Since the present value of an annuity of £1 during the life of A may be represented by $S(a) = \frac{1}{a}(a_1 p + a_2 p^2 + a_3 p^3 + \text{\&c.})$; and the present value of a life one year older may be represented by $S(a_1) = \frac{1}{a_1}(a_2 p + a_3 p^2 + a_4 p^3 + \text{\&c.})$ it is obvious that $1 + S(a_1) = \frac{1}{a_1}(a_1 + a_2 p + a_3 p^2 + \text{\&c.})$,

$$\text{consequently } S(a) = \frac{a_1 p}{a}(1 + S(a_1)).$$

This is a remarkably simple formula for calculating the present value of annuities on single lives, beginning with the oldest in the table of mortality.

According to the Northampton Tables of Mortality, the value of an annuity upon a life of 96 is $= 0$: now if the rate is 5 per cent. we have for

95 years	(1 + 0)	× $\frac{1}{4}$	× ·95238	=	·2381
94 „	(1 + ·2381)	× $\frac{4}{9}$	× ·95238	=	·5241
93 „	(1 + ·5241)	× $\frac{9}{16}$	× ·95238	=	·8165

92	„	$(1 + \cdot 8165) \times \frac{16}{24}$	$\times \cdot 95238$	$=$	$1 \cdot 1533$
91	„	$(1 + 1 \cdot 1533) \times \frac{24}{34}$	$\times \cdot 95238$	$=$	$1 \cdot 4476$
90	„	$(1 + 1 \cdot 4476) \times \frac{34}{46}$	$\times \cdot 95238$	$=$	$1 \cdot 7229$
89	„	$(1 + 1 \cdot 7229) \times \frac{46}{62}$	$\times \cdot 95238$	$=$	$1 \cdot 9240$
88	„	$(1 + 1 \cdot 9240) \times \frac{62}{83}$	$\times \cdot 95238$	$=$	$2 \cdot 0802$
87	„	$(1 + 2 \cdot 0802) \times \frac{83}{111}$	$\times \cdot 95238$	$=$	$2 \cdot 1935$
86	„	$(1 + 2 \cdot 1935) \times \frac{111}{145}$	$\times \cdot 95238$	$=$	$2 \cdot 3283$
85	„	$(1 + 2 \cdot 3283) \times \frac{145}{186}$	$\times \cdot 95238$	$=$	$2 \cdot 4711$
&c.		&c.	&c.		&c.

Prob. IV. To find the present value of an annuity of £1 upon two joint lives A and B.

If $S(a)$ be the present value of an annuity of £1 on the life of A, and $S(b)$ the value upon the life of B, then $S(ab)$ is the value of the two joint lives $= r_1 s_1 p + r_2 s_2 p^2 + r_3 s_3 p^3 + \&c.$ to the end of the table.

If the present value of two joint lives, each respectively one year older than A and B, be represented by $S(a_1 b_1)$, then it may be proved similarly to the last problem that

$$S(ab) = \frac{a_1 b_1 p}{ab} (1 + S(a_1 b_1)).$$

Beginning our calculation then with the oldest, and taking 10 years as the difference of the ages, we will have, for two lives, the one 96 and the other 86, the value $= 0$; then

95 and 85	$=$	$(1 + 0)$	$\times \frac{1}{4}$	$\times \frac{145}{186}$	$\times \cdot 95238$	$=$	£	$\cdot 1856$
94 „ 84	$=$	$(1 + \cdot 1856)$	$\times \frac{4}{9}$	$\times \frac{186}{234}$	$\times \cdot 95238$	$=$		$\cdot 3989$
93 „ 83	$=$	$(1 + \cdot 3989)$	$\times \frac{9}{16}$	$\times \frac{234}{289}$	$\times .95238$	$=$		$\cdot 6068$
92 „ 82	$=$	$(1 + \cdot 6068)$	$\times \frac{16}{24}$	$\times \frac{289}{346}$	$\times \cdot 95238$	$=$		$\cdot 8521$
91 „ 81	$=$	$(1 + \cdot 8521)$	$\times \frac{24}{34}$	$\times \frac{346}{406}$	$\times \cdot 95238$	$=$		$1 \cdot 0611$
90 „ 80	$=$	$(1 + 1 \cdot 0611)$	$\times \frac{34}{46}$	$\times \frac{406}{469}$	$\times \cdot 95238$	$=$		$1 \cdot 2556$
&c.		&c.		&c.				&c.

Prob. V. To find the present value of an annuity of £1 on the three joint lives A, B, C.

If $S(a)$ be the present value of the annuity on the life A, $S(b)$ that on the life B, and $S(c)$ that on the life C; then the present value of the three joint lives is $S(abc) = r_1 s_1 t_1 p + r_2 s_2 t_2 p^2 + r_3 s_3 t_3 p^3 + \&c.$ to the end of the table. Now, if the present value of three joint lives, each respectively one year older than A, B, C, be represented by $S(a_1 b_1 c_1)$, it may be proved, as in Prob. III., that

$$S(abc) = \frac{a_1 b_1 c_1 p}{abc} (1 + S(a_1 b_1 c_1)).$$

Beginning with the oldest life, and taking 10 years as the difference between each, we have, at 5 per cent.,

	Ages.								
75	85	95	$= (1 + 0)$	$\times \frac{752}{832}$	$\times \frac{145}{186}$	$\times \frac{1}{4}$	$\times \cdot 95238 =$	£	$\cdot 1678$
74	84	94	$= (1 + \cdot 1678)$	$\times \frac{832}{912}$	$\times \frac{186}{234}$	$\times \frac{4}{9}$	$\times \cdot 95238 =$		$\cdot 3585$
73	83	93	$= (1 + \cdot 3585)$	$\times \frac{912}{992}$	$\times \frac{234}{289}$	$\times \frac{9}{13}$	$\times \cdot 95238 =$		$\cdot 5417$
72	82	92	$= (1 + \cdot 5417)$	$\times \frac{992}{1072}$	$\times \frac{289}{346}$	$\times \frac{16}{24}$	$\times \cdot 95238 =$		$\cdot 7566$
71	81	91	$= (1 + \cdot 7566)$	$\times \frac{1072}{1152}$	$\times \frac{346}{406}$	$\times \frac{24}{34}$	$\times \cdot 95238 =$		$\cdot 9365$
70	80	90	$= (1 + \cdot 9365)$	$\times \frac{1152}{1232}$	$\times \frac{406}{469}$	$\times \frac{34}{46}$	$\times \cdot 95238 =$		$1 \cdot 1034$
		&c.	&c.		&c.		&c.		

PROB. VI. To find the present value of an annuity of £1 on the longest of two joint lives, A and B.

It has been already shown, that the probability that at least one of them shall be alive at the end of one year is $r_1 + s_1 - r_1 s_1$, and therefore the present value of £1 to be paid on this contingency is $p(r_1 + s_1 - r_1 s_1)$. The present value of £1 to be paid at the end of the second year upon the same contingency is $p^2(r_2 + s_2 - r_2 s_2)$, and at the end of three years it is $p^3(r_3 + s_3 - r_3 s_3)$, and so on. The present value of all the payments is consequently $p(r_1 + s_1 - r_1 s_1) + p^2(r_2 + s_2 - r_2 s_2) + p^3(r_3 + s_3 - r_3 s_3) +$ &c. Now the sum of all these is $(r_1 p + r_2 p^2 +$ &c.$) + (s_1 p + s_2 p^2 +$ &c.$) - (r_1 s_1 p + r_2 s_2 p^2 +$ &c.$)$ and from what has formerly been shown, this sum is obviously $= S(a) + S(b) - S(ab)$; that is,

From the sum of the present values of the single lives take the present value of the joint lives, and the remainder is the present value of the longest of the two joint lives.

PROB. VII. To find the present value of an annuity of £1 on the longest of three lives A, B, and C.

The probability that at least one of the three shall be alive at the end of one year has been shown to be $1 - (1 - r_1)(1 - s_1)(1 - t_1)$, or $(r_1 + s_1 + t_1) - (r_1 s_1 + r_1 t_1 + s_1 t_1) + r_1 s_1 t_1$, and therefore the present value of £1 to be paid at the end of the first year on this contingency is $p((r_1 + s_1 + t_1) - (r_1 s_1 + r_1 t_1 + s_1 t_1) + r_1 s_1 t_1)$. The present value of £1 to be paid at the end of the second year is $p^2((r_2 + s_2 + t_2) - (r_2 s_2 + r_2 t_2 + s_2 t_2) + r_2 s_2 t_2)$, and so on; therefore the sum of these, or $(r_1 p + r_2 p^2 +$ &c.$) + (s_1 p + s_2 p^2 +$ &c.$) + (t_1 p + t_2 p^2 +$ &c.$) - (r_1 s_1 p + r_2 s_2 p^2 +$ &c.$) - (r_1 t_1 p + r_2 t_2 p^2 +$ &c.$) - (s_1 t_1 p + s_2 t_2 p^2 +$ &c.$) + (r_1 s_1 t_1 p + r_2 s_2 t_2 p^2$

$+ \&c.)$ is manifestly equal to $(S(a)+S(b)+S(c))-(S(ab)+S(ac)+S(bc))+S(abc))$; that is,

From the sum of the present values of the single lives subtract the sum of the values of the joint lives taken two by two, and to the remainder add the value of the three joint lives; the result is the value of the longest of the three joint lives.

PROB. VIII. To find the value of an annuity of £1 on the life of A, which shall not commence until the expiration of v years.

The value of this annuity, from what has been already proved, may be shown to be

$$\frac{1}{a}(a_{v+1}p^{v+1}+a_{v+2}p^{v+2}+a_{v+3}p^{v+3}+\&c.)$$

$$=\frac{a_v}{a}p^v\times\frac{1}{a_v}(a_{v+1}p+a_{v+2}p^2+a_{v+3}p^3+\&c.)$$

$$=r_v p^v\,S(a_v);\text{ that is,}$$

Find the value of an annuity on a life v years older than the given life, multiply this by the probability that the given life will continue v years, and this product multiplied by the present value of £1 payable in v years is the value required.

If we make $S(a, v)$ to represent the value of an annuity on the life of A for v years, then it is manifest that

$$S(a, v)=S(a)-r_v p^v S(a_v);\text{ that is,}$$

The difference of the value of an annuity on the given life, and the value of the *deferred* annuity, as found above, is the value of the *temporary* annuity.

From these principles it may be readily inferred, that the value of an annuity on the joint lives of A and B, deferred for v years, is

$$r_v s_v p^v\,S(a_v b_v),$$

and consequently the temporary annuity on the joint lives for v years is

$$S(ab)-r_v s_v p^v\,S(a_v b_v).$$

MISCELLANEOUS QUESTIONS.

1. What number is that to which if 56 be added the sum will be thrice the number? Ans. 28.

2. Divide the number 92 into four parts such, that the first shall exceed the second by 10, the third by 18, and the fourth by 24. Ans. 36, 26, 18, and 12.

3. When the hour and minute hands of a watch are exactly together between six and seven o'clock, what is the exact time? Ans. $32\frac{8}{11}$ minutes past six.

4. Divide £200 between A and B, so that if A's share be divided by B's the quotient shall be 3.

Ans. A's share 150, and B's 50.

5. Divide £56 between A and B, so that A's share shall be to B's as 5 to 2. Ans. A 40, and B 16.

6. A person having a certain number of eggs, left one half the number and half an egg at one place; half the remainder and half an egg at a second place; half the remainder and half an egg at a third place, without breaking any, and then he had one egg left. How many had he at first? Ans. 15.

7. After drawing 34 gallons out of one of two equal casks, and 80 gallons out of the other, there remained just twice as much liquor in the one as in the other. What was the content of the casks? Ans. 126 gallons.

8. A hare, 50 of her own leaps before a greyhound, makes 4 leaps for the dog's 3; but 2 leaps of the dog are equal to 3 of the hare's leaps. How many leaps must the dog make to catch the hare? Ans. 300.

9. A waterman can row 5 miles *with* the tide in three quarters of an hour; but it takes him one hour and a half to row the same distance against the tide when it is only half as strong. What is the velocity of the strongest tide?

Ans. $2\frac{2}{9}$ miles an hour.

10. A person in a party at cards bet three shillings to two upon every deal, and after twenty deals found he had gained 5s. How many deals did he win? Ans. 13.

11. A bill of £120 was paid in guineas and moidores, the number of pieces of both sorts being 100. How many were there of each? Ans. 50 of each.

12. A farmer sold to one person 30 bushels of wheat and 40 of barley for £13, 10s.; and to another he sold 50 bushels of wheat and 30 of barley for £17, which was at the same rate per bushel as the former. What was the price of a bushel of each?

Ans. The wheat was 5s. and the barley 3s. the bushel.

13. A gentleman hired a servant for 12 months, and agreed to give him £20 and a livery; but the servant left at the

end of 8 months, and received £12 and the livery as his wages for that time. What was the value of the livery?

Ans. £4.

14. A man and his wife usually drank out a vessel of beer in 12 days, and it was found when the man was alone that it served him 20 days. How long would it serve the wife?

Ans. 30 days.

15. Four men having found a purse filled with shillings only, each took a number at an adventure, and on comparing their number afterwards it was found that if the first was to take 25s. from the second, he would then have as much as the second had left; if the second was to take 30s. from the third, he would then have triple of what the third had left; if the third was to take 40s. from the fourth, he would then have double of what the fourth had left; and if the fourth was to take 50s. from the first, he would then have thrice as much as the first had left and 5s. more. What did each take?

Ans. The first 100s., second 150s., third 90s., and fourth 105s.

16. Several merchants enter into company; each puts into the stock 65 times as many pounds as there were partners, and with that stock they traded, and gained as many pounds per £100 as there were partners. Now, if £10, 10s. be added to and subtracted from their gain, the product of their sum and difference will be £6491 : 6 : 3. How many partners were there?

Ans. 5.

17. A company of foot are 1165 of their own paces before a troop of horse, and take 5 paces in the time that the horse take 4, but 3 paces of the horse are equal to 4 of the foot. How many paces must the horse march before they overtake the foot?

Ans. 13980.

18. To find a number such, that if it be divided into three equal parts, and also into four equal parts, the continued product of the former shall be equal to the continued product of the latter.

Ans. $9\frac{13}{27}$.

19. A spirit-merchant mixes brandy worth 36s. per gallon, with whisky worth 10s. per gallon, in such proportions, that he may gain 30 per cent. by selling the mixture at 30s. per gallon. What quantity must he take of each?

Ans. 85 gallons brandy, and 84 gallons spirits.

20. A and B, employed together on the same work, can earn £2 in 6 days; A and C together can earn £2, 14s. in 9 days; and B and C together can earn £4 in 15 days. What can each separately earn in one day?

Ans. A 3s. 8d., B 3s., and C 2s. 4d.

21. To divide 90 into four parts such, that the first increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, shall all be equal to the same number. Ans. 18, 22, 10, and 40.

22. Find five numbers such, that the first with $\frac{1}{3}$ the sum of the other four is equal to 2380, the second with $\frac{1}{4}$ the sum of the others = 2205, the third with $\frac{1}{5}$ the sum of the others = 2268, the fourth with $\frac{1}{6}$ the sum of the others = 2450, and the fifth with $\frac{1}{7}$ the sum of the others = 2700.

Ans. 420, 840, 1260, 1680, and 2100.

23. A person being asked his age, answered, if $\frac{1}{80}$ of the years I have lived were multiplied by $\frac{5}{8}$ of them, the product would be my age. How old was he? Ans. 96 years.

24. What two numbers are those whose sum is 1, and whose quotient is 100000. Ans. $\frac{1}{100001}$ and $\frac{100000}{100001}$.

25. Two ships, laden with the same kind of wine, paid shore-dues in proportion to the quantity of wine they had on board. Now the first had on board 250 hhds., and paid 1 hhd., and also 36 shillings; the second had on board 400 hhds., and paid 2 hhds., and got back 20 shillings. What was the value of the hhd. of wine? Ans. £4, 14s.

26. A merchant bought 40 bushels of wheat, 24 of barley, and 20 of oats, for £15, 12s.; he afterwards bought at the same price, for each sort of grain, 26 bushels of wheat, 30 of barley, and 50 of oats, for £16; and at a third time he bought 24 bushels of wheat, 120 of barley, and 100 of oats, for £34. What did each kind of grain cost him per bushel?

Ans. The wheat 5s., the barley 3s., and the oats 2s. per bushel.

27. A set out from London for Bristol on the same day that B set out from Bristol for London; when they met, A said to B, I have travelled 20 miles more than you, and have gone as many miles in $6\frac{2}{3}$ days as you have gone miles in all hitherto. But said B, at the end of 15 days from this, I shall be in London if I travel still at the same rate. How many miles had each travelled when they met?

Ans. A had gone 60 miles, and B 40 miles.

28. A wall containing 18,225 cubic feet is 5 times as high as it is broad, and 8 times as long as it is high. What is its breadth? Ans. $4\frac{1}{2}$ feet.

29. Find three numbers such, that the sum of the first and second multiplied by the third may be 63, the sum of the

second and third multiplied by the first may be 28, and the sum of the first and third multiplied by the second may be 55.

Ans. 2, 5, and 9.

30. There is a number consisting of two digits whose square root is equal to the quotient of the number divided by 4 times the difference of the digits, and the sum of the digits is equal to 5 times their difference. What is the number?

Ans. 64.

31. What number is that consisting of two digits, the square of whose sum is the number itself, and if the square of the difference of the digits be taken from the number, the remainder is 4 times the left hand digit? Ans. 81.

32. What two numbers are those whose product is 100, and the difference of their square roots is 3? Ans. 4 and 25.

33. What two numbers are those whose difference is 15, and the half of whose product is equal to the cube of the less?

Ans. 3 and 18.

34. A manufacturer introduces a new branch of business into a country, and takes 5 apprentices, who are bound for 7 years; after which time he takes 5 more; and each of his former apprentices takes 5 also; and these in their turn, along with himself and the former apprentices, take 5 more; and so on. Now, supposing none to die, how many will be bred to the business in 42 years? Ans. 46656.

35. The duties on certain goods amounted to £2460, out of which a discount was allowed of $2\frac{1}{2}$ per cent. upon the sum actually paid for prompt payment. What did the discount amount to? Ans. £60.

36. A merchant discounted two bills at the bank, one of them for £550, payable in 7 months, and the other for £720, payable in 4 months; and he received for the whole £1200. At what rate per cent. per annum was the interest charged?

Ans. £13·267 per cent. per annum.

37. The common difference of four numbers in arithmetical progression is 4, and their continued product is 21945. What are the numbers? Ans. 7, 11, 15, and 19.

38. The sum of ten numbers in arithmetical progression is 120, and the sum of their cubes is 29160. What are the numbers? Ans. 3, 5, 7, 9, &c.

39. Given the sum of the numbers 0, 1, 2, 3, &c. = 1225, to find the sum of their squares? Ans. 40425.

40. Two persons set out at the same time, from two places 462 miles distant, to meet one another. The first goes 1

mile the first day, 2 the second, and so on. The other travels each day the cubes of the number of miles that the first travelled on that day. In what time will they meet?

Ans. 6 days.

41. A gentleman sold an estate for the value of the trees upon it above 7 feet in circumference, at one pound for the first, two for the second, four for the third, and so on, doubling the price of each successive tree. The value of the estate came to £65,535. How many trees of the above description were upon it? Ans. 16 trees.

42. A gentleman had seven children, whose ages differed successively by one year. In giving them new clothes, he determined to bestow as many yards of lace on the trimming of the youngest as he was years old, on the second as many as the sum of the ages of the two youngest, on the third as many as the sum of the ages of the three youngest, and so on; and he agreed to pay the tailor for making each suit the product in pence of the child's age by the number of yards of lace on his suit. The bill amounted to £7 : 10 : 6. What were the ages of the children? Ans. The youngest was 5 years.

43. A merchant discounted two bills; the first had six months to run, and the other 8 months. The value of both came to £308 : 6 : 8, and the discount to £8 : 6 : 8. Had interest been charged upon the bills instead of discount, it would have come to 4s. $8\frac{2}{3}$d. more than the discount. Required the value of the bills?

Ans. The first was £205, and the other £103 : 6 : 8.

44. If £400 be the present value of an annuity to continue 23 years after the expiration of 8 years, what would be its value for 21 years after the expiration of 10 years, interest at 5 per cent.? Ans. £344·8624.

45. A gentleman had 10 different annuities of £100 each; their continuance differed by one year each, and the longest was for 60 years. He sold them all at 5 per cent. compound interest. What money did he receive for them?

Ans. £18653·2142.

46. A bookseller purchases a work for £40, and pays for printing 1000 copies of it £15, for paper £20, and for incidents £10. He sells the edition in 10 years at 3s. each copy. How much does he gain per cent. per annum.

Ans. £11, 19s. per cent. per annum.

47. A person who owes his creditor £320 just now, and £96 more at the end of five years, wishes to pay the whole in

one payment. What is the proper time for doing this, according to the true principle of equation of payments, namely, that the simple interest shall be equal to the discount?

Ans. At the end of one year.

48. A usurer lent £186 for a certain time, and gained £31; and by lending £360 at the same rate for another time, he gained £90. The sums of the times they were lent amounted to 20 months. How long time was each sum lent?

Ans. The first 8 months, the other 12 months.

49. A vessel, containing 200 cubical inches, is filled with a mixture of wine and spirits, the specific gravity of the wine is ·950, of the spirits ·858, and of the mixture ·881. How much of each does the vessel contain?

Ans. 50 cubic inches of wine, and 150 of spirits.

50. Paid the sum of £30 in half-guineas and half-crowns only, and the number of pieces was 80. What was the number of each? Ans. 50 half-guineas, and 30 half-crowns.

51. In how many ways may £100 be paid in guineas, crowns, and moidores, the number of pieces being 100?

Ans. 9 different ways.

52. How many bushels of corn at 42d. and 48d. must be mixed with 72 bushels at 60d. so that a bushel of the mixture may be worth 54d.? Ans. At 42d. 1, 2, 3, &c. to 35.
At 48d. 70, 68, 66, &c. to 2.

53. What two numbers are those which are to one another in the ratio of 3 to 5, and whose squares added together make 1666? Ans. 21 and 35.

54. Divide £16 between two persons, so that the difference of the cubes of their shares may be 386.

Ans. £9 and £7.

55. What two numbers are those, whose product multiplied by the greater will produce 405, and whose difference multiplied by the less will produce 20? Ans. 5 and 9.

56. What two numbers are those, of which the sum of their squares is 208, and the sum of their cubes is 2240?

Ans. 12 and 8.

57. To find a sum of money in pounds and shillings whose half is just its reverse.* Ans. £13, 6s.

58. An oblong pond measuring 15,000 square yards was surrounded by a walk 7 yards broad, and measuring 3696

* The reverse of a sum of money such as £3, 10s., is £10, 3s.

square yards. What was the length and breadth of the pond? Ans. 150 yards long and 100 broad.

59. Required a number consisting of two digits such, that when divided by the product of its digits the quotient is 3, and if 18 be added to it the digits are reversed. Ans. 24.

60. A traveller sets out from one city A, to go to another B, at the same time that another traveller sets out from B to go to A; they both travel uniformly at such rates, that the former 4 hours after their meeting arrives at B, and the latter in 9 hours after at A. In what time did each perform the journey?

Ans. The former in 10 hours and the latter in 15 hours.

61. A sets out on a journey, and goes 8 miles the first day, 12 the second, and 16 the third, and so on; at the same time B sets out and goes the same way, 1 mile the first day, 4 the second, 9 the third, and so on, according to the square of the number of days. In how many days will he overtake A?

Ans. In 7 days.

62. Bought 12 loaves for 12d., some of the loaves were 2d., others 1d., and the rest one farthing each. How many were there of each sort?

Ans. 3 at 2d., 5 at 1d., and 4 at one farthing.

63. In how many different ways is it possible to pay £1000 without using any other coin than moidores, guineas, and crowns? Ans. In 70,734 different ways.

64. Required the diameter of a globe of which the superficial and solid contents are both expressed by the same number? Ans. 6.

65. A board is 10 feet long, 8 inches in breadth at the one end, and 6 inches at the other. How much must be cut off from the less end to make a square foot?

Ans. 23·2493 inches.

66. A garden is 100 yards long, and 80 yards broad, and a border of equal breadth surrounds the sides of it, which is just one-half the content of the garden. What is its breadth?

Ans. 12·9844 yards.

67. A person received a box of oranges, containing between 100 and 200, and on counting them over by 2, 3, 4, 5, and 6 at a time, he found there were none over; but when he told them over by 7 at a time, there were 5 remaining. How many were in the box. Ans. 180.

68. A gentleman by accident broke a basket of eggs, and offered at once to pay for them if the owner could tell him

the number that were in the basket. The owner said there were between 300 and 400, and that when they were reckoned by 2, 3, or 5 at a time there always remained 1, but when he reckoned them by 7 at a time none remained. What number of eggs were in the basket? Ans. 301.

69. Given the sum 170, and the product 94864, of the sides of an isosceles triangle, to find the sides.

Ans. The sides each 77, and the base 16.

70. A tobacconist has three kinds of tobacco; one at 2s. 8d. per pound, another at 20d., and a third at 16d. He wishes to make a mixture of 56 lbs. of these worth 22d. per pound. How many pounds of each sort must he take?

Ans. Of the first, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, or 20.
Of the second, 44, 40, 36, 32, 28, 24, 20, 16, 12, 8, or 4.
Of the third, 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, or 32.

71. Given $x^{\frac{1}{x}} = 1{\cdot}2655$, to find an approximate value of x.

Ans. $x = 1{\cdot}3876$.

72. Two joiners are employed at different wages; the first wrought a certain number of days and received 96s., the other wrought 6 days less and received 54s. If the second had wrought the whole time, and the first 6 days less, their wages would have been equal. Required how many days they wrought, and their daily wages?

Ans. The first worked 24 days at 4s. a-day, and the second 18 days at 3s. a-day.

73. To find four square numbers such, that their sum added to the sum of their roots shall make 12.

Ans. $\frac{361}{100}$, $\frac{169}{100}$, $\frac{121}{100}$, and $\frac{49}{100}$.

74. To find four square numbers such, that their sum diminished by the sum of their roots shall leave 4.

Ans. $\frac{441}{100}$, $\frac{289}{100}$, $\frac{169}{100}$, and $\frac{121}{100}$.

75. To find two numbers such, that their product with their sum either added to, or subtracted from it, shall make a cube? Ans. $\frac{27}{8}$, and $\frac{91}{64}$.

76. In how many different ways is it possible to pay £20,000 with guineas and crowns only?

Ans. 190 different ways.

77. To find three numbers such, that their sum shall be 2,

and that the sum of their cubes diminished by any one of them shall be a square. Ans. $\frac{97751}{149769}$, $\frac{97751}{149769}$, and $\frac{104036}{149769}$.

78. A merchant bought two pipes of wine, one consisting of measures of 5 drams, and the other of measures of 8 drams, and for the whole paid a price which was a square number of pieces of money, to which if 60 be added, the sum will be a square, of which the root is equal to the whole number of measures. How many measures were there of each?

Ans. $\frac{79}{12}$ the number of measures of 5 drams, and $\frac{59}{12}$ the number of measures of 8 drams.

79. To find a right-angled triangle of which the hypothenuse lessened by either side shall leave a cube?

Ans. 13, 12, and 5 are the sides.

80. To find a right-angled triangle of which the sum of the hypothenuse and either of the sides shall make a square.

Ans. $\frac{377}{64}$, $\frac{133}{64}$, and $\frac{352}{64}$ are the sides.

81. To find a right-angled triangle of which the difference of the sides about the right angle, as also the greater of them, shall be each a square, and likewise the sum of the area and the smaller side shall be a square.

Ans. The two sides 484 and 363, and the hypothenuse 605.

82. To find a right-angled triangle such, that the sum of the area and hypothenuse shall be a square, and that the sum of the three sides shall be a cube.

Ans. The three sides are 2, $\frac{621}{50}$, and $\frac{629}{50}$.

83. To find a right-angled triangle such, that the square of the hypothenuse shall be the sum of a square and its root, and if it be divided by one of the sides about the right angle, the quotient shall be the sum of a cube and its root.

Ans. The three sides are $\frac{12}{9}$, $\frac{16}{9}$, and $\frac{20}{9}$.

84. Required three numbers in continued proportion such, that their three differences shall be all squares?

Ans. 567, 1008, and 1792.

85. Given $x^y = 5000$, and $y^x = 3000$, to find approximate values of x and y.

Ans. $x = 4{\cdot}691445$, and $y = 5{\cdot}510132$.

86. To find the sides of three right-angled triangles that shall have equal areas.

Ans. 74, 24, and 70; 58, 40, and 42; 113, 15, and 112.

87. Divide 100 into three parts such, that half the first plus $\frac{1}{4}$ of the third plus 1; $\frac{2}{3}$ of the second minus 1 plus half the first; and $\frac{3}{4}$ of the third minus 1 plus $\frac{1}{3}$ of the second, shall be all equal to each other.

Ans. $\frac{922}{21}$, $\frac{390}{21}$, and $\frac{788}{21}$.

88. To find three numbers such, that the product of any two of them added to the sum of the same two, shall make a square. Ans. 4, 9, and 28.

89. To find three numbers such, that the product of any two of them diminished by the third shall be a square.

Ans. $\frac{5}{4}$, $\frac{13}{4}$, and $\frac{20}{4}$.

90. To find two numbers such, that the square of their sum taken from either of them shall leave a square.

Ans. $\frac{10}{225}$ and $\frac{5}{225}$.

91. To find two numbers such, that if a given square (4) be added to each, and also to their sum and difference, the four results will be squares. Ans. 12096 and 22496.

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